Study of symmetry in non-monotonic logics

Belaïd Benhamou, Tarek Nabhani and **Pierre Siegel**
Université de Provence
Laboratoire des Sciences de l'Information et des Systèmes (LSIS)
Centre de Mathématiques et d'Informatique
39, rue Joliot Curie - 13453 Marseille cedex 13, France
Email: {Belaid.Benhamou;Tarek.Nabhani;siegel}@cmi.univ-mrs.fr

Abstract

Symmetry had been well studied in classical logics and constraint programming since a decade. Early, Krishnamurthy showed that some tricky formulas admit short proofs when augmenting the propositional logic resolution proof system by the symmetry rule. However, in Artificial Intelligence, we usually manipulate incomplete information and need to include uncertainty to reason on knowledge with exceptions and non-monotonicity. Several non classic logics are introduced for that purpose, but as far as we know, symmetry for these frameworks had not been studied yet. Here, we are interested to extend the notion of symmetry to that non classical logics such as Preferential logics, X-logics and Default logics, then give new symmetry inference rules for the X-logics and the Default logics that can be used to shorten proofs. Finally, we show how symmetry reasoning is profitable for these logics and how they handle some symmetries that do not exist in classical logics.

Introduction

Symmetry is by definition a multidisciplinary concept. It appears in many fields ranging from mathematics to artificial intelligence, chemistry and physics. In general, it returns to a transformation, which leaves invariant (does not modify its fundamental structure and/or its properties) an object (a figure, a molecule, a physical system, a formula or a constraints network...). For instance, rotating a chessboard up to 180 degrees gives a board that is indistinguishable from the original one. Symmetry is a fundamental property that can be used to study these various objects, to finely analyze these complex systems or to reduce the computational complexity when dealing with combinatorial problems.

As far as we know the principle of symmetry in AI has been first introduced by Krishnamurthy (Krishnamurty 1985) to improve resolution in propositional logic. Symmetries for Boolean constraints are studied in depth in (Benhamou and Sais 1992; 1994; Benhamou, Sais, and Siegel 1994). The authors showed how to detect them and proved that their exploitation is a real improvement for several automated deduction algorithms efficiency. Since that, many research works on symmetry appeared. For instance, the static approach used by James Crawford et al. in (Crawford et al. 1996) for propositional logic theories consists in adding constraints expressing global symmetry of the problem. This technique has been improved in (Aloul et al. 2003) and extended to 0-1 Integer Logic Programming in (Aloul et al. 2004). The notion of interchangeability in Constraint Satisfaction Problems (CSPs) is introduced in (Freuder 1991) and symmetry for CSPs is studied earlier in (Puget 1993; Benhamou 1994).

Since a great number of constraints could be added in the static approach, in CSPs, some researchers proposed to add the constraints during the search. In (Backofen and Will 1999; Gent and Smith 2000; Gent, Harvey, and Kelsey 2002), authors post some conditional constraints which remove the symmetric of the partial interpretation in case of backtracking. In (Focacci and Milano 2001; Fahle, Schamberger, and Sellmann 2001; Puget 2002; Gent et al. 2003), authors proposed to use each subtree as a no-good to avoid exploration of some symmetric interpretations and the group equivalence tree conceptual for value symmetry elimination is introduced in (Roney-Dougal et al. 2004). More recently Walsh in (Walsh 2006) studied various new propagators to break various symmetries among them the one acting simultaneously on both variables and values.

As stated by Krishnamurthy in his seminal work (Krishnamurty 1985), symmetry is one of the most promising approach for deriving short proofs for many tricky formulas. He proposed a resolution proof system augmented with a local symmetry rule (LS-Res) that Stephan Szeider in a more recent work (Szeider 2005) showed to be stronger than resolution with the global symmetry rule which is itself stronger than the classical resolution proof system.

On other hand, within the framework of the Artificial intelligence, an important paradigm is to take into account incomplete information (uncertain information, revisable information...). An essential component of the intelligence (that is human, animal or artificial) is indeed to be related to a certain capacity of adaptation of the reasoning. Contrary to the mode of reasoning formalized by a conventional or a classical logic, a result deducible from information (from a knowledge, or from beliefs) is not true but only probable in the sense that it can be invalidated further, and can be revised when adding new information. For example, it is admitted that a normal bird flies. Thus, if it is known that Tweety is a bird, then one will conclude from it that naturally Tweety flies. If it is learned thereafter that Tweety is a Penguin, this conclusion will have to be revised. This is impossible in a classical logic having the monotony property: an information deducible from a knowledge C, it will be always if C is increased.

To manage the problem of exceptions, several logical approaches in Artificial intelligence had been introduced. Many non-monotonic formalisms were presented since about thirty years, but the problem of symmetry within this framework was not studied. Symmetry reasoning is however relevant for knowledge representation and non-monotonic reasoning. For instance, in the previous example, it is interesting to consider that the normal birds belong to the same class with respect to some basic properties, and then they are all symmetrical in this sense.

In this work, we investigate symmetry in three non-classical logics: Preferential logic (Bossu and Siegel 1982; 1985; Shoam 1987; Besnard and Siegel 1988; Kraus, Lehmann, and Magidor 1990), X-logic formalism (Siegel, Forget, and Risch 2001), and Default logic (Reiter 1980).

The motivation behind symmetry in non-monotonic reasoning is to find short proofs for theorem provers that deal with these logics. For instance in each of the X-logic and the default logic, we introduced a symmetry rule that can be used to infer all the symmetrical formulas of a deduced formula without duplication of efforts. That is, if a formula is proved to be a theorem in these logics, then we can deduce directly by symmetry that all its symmetrical formulas should be too. Otherwise, if the checked formula is not a theorem, then we conclude by symmetry that all its symmetrical formulas are not theorems. Thus, such symmetry rules make cuts in the proof and help a theorem prover to find a short one.

The rest of the paper is organized as follows: Section 2 gives the main definitions of symmetry in propositional logic. In section 3, we study symmetry in preferential logics. Section 4 extends symmetry to the X-logic formalism. We introduce in Section 5 symmetry in Defaults logic. Finally, Section 6 concludes the work and gives some perspectives.

Symmetry in Propositional Logic

First, we give the semantic symmetry definition in Propositional logic:

Definition 1 (Semantic symmetry) *Let F be a propositional formula given in CNF and L_F its complete set* [1] *of literals. A semantic symmetry of F is a permutation σ defined on L_F such that $F \models \sigma(F)$ and $\sigma(F) \models F$*

In other words a semantic symmetry of a formula is a variable permutation that conserves the set of the models of the formula. It also conserves the set of no-goods (counter models). Now, we recall the syntactic symmetry definition (Benhamou and Sais 1992; 1994).

Definition 2 (Syntactic symmetry) *Let F be a propositional formulas given in CNF and L_F its complete set of literals. A syntactic symmetry of F is a permutation σ defined on L_F such that the following conditions hold:*

1. $\forall l \in L_F, \sigma(\neg l) = \neg\sigma(l)$,
2. $\sigma(F) = F$

In other words, a syntactical symmetry of a formula is a variable permutation that leaves the formula invariant. It is trivial to see that each syntactic symmetry is a semantic symmetry, and in general, the converse is not verified. It is known

[1]The set of literals of F containing each variable of F and its negation

On other hand, Krishnamurthy introduced the following symmetry rule to augment the resolution proof system.

Proposition 1 *If L is the propositional logic, A a set of formulas of L, B a formula of L and σ a syntactic symmetry of A, then the symmetry rule can defined as follows:*

$$\frac{A \vdash B}{A \vdash \sigma(B)}$$

Many hard problems for resolution have been shown to be polynomial when using symmetry in resolution. For instance, finding some of the Ramsey's numbers or solving the pigeon-hole problem are known to be exponential for classical resolution, while short proofs can be made for both them when adding the symmetry rule to the resolution proof system. We will see in Section 4 how to extend this rule to non-monotonic logics.

Now we deal with the main contribution of this work that consists in extending symmetry to non-monotonic logics.

Symmetry in Preferential Logic

Symmetry is very important in non-monotonic reasoning, but it is not investigated until now. Here we extend the notion of symmetry to the preferential logic framework.

For example, it is admitted that a normal student is young. Thus, if it is known that John is a student, one will conclude from it rather naturally that John is young. If it is learned thereafter that John is fifty years old, then the conclusion saying that John is young will be revised.

It is then important to consider that the normal students belong to the same class, since they are all symmetrical with respect to the *normal* property.

Initially, simplest is to start from a preferential approach, such as it was initiated by Bossu-Siegel (Bossu and Siegel 1982; 1985), taken again by Shoam (Shoam 1987), and Besnard-Siegel (Besnard and Siegel 1988) , then by Kraus, Lehmann and Magidor in (Kraus, Lehmann, and Magidor 1990). All these approaches are build on a classical logic (propositional calculus, predicate calculus, modal logic) where the semantic of the inference is given by "A formula A implies a formula B if each model of A is a model of B". However, a preferential approach, in its most general form, says "A implies B if all the preferred models of A are models of B". The preferred models of A are models that have relevant properties for the management of the exceptions. This concept of preference can be defined by a relation of pre-order (a transitive and reflexive relation) between interpretations, the preferred models being the minimal models for this relation. For our elementary example, if I and J are interpretations and the pertinent information is "Young", then the pre-order relation can be defined by: $I \prec J$ iff any young individual in J is young in I

Definition 3 *Let L be a classical logic, and F the set of all formulas of L. If A is a subset of formulas (or a formula) of F, then $\bar{A}$ is the set is of the formulas logically implied by A. The set of formula A is deductively closed if $A = \bar{A}$.*

Definition 4 *A preferential relation $\prec$ is any pre-order relation between interpretations (a pre-order relation $\prec$ is transitive and reflexive). Besides, if the relation $\prec$ is antisymmetric, then $\prec$ becomes an order.*

Intuitively, we can consider the students who are not young as exceptions (Abnormal students). Therefore $I \prec J$ if the set of exceptions of I is include in the set of exceptions of J.

Definition 5 *If A is a set of formulas, a minimal model M of A is an interpretation which satisfies A ($M \vdash A$) and which is minimal for the relation $\prec$ defined on the set of models of A. That is, if M' is a model of A such that $M' \prec M$, then $M \prec M'$ (or, equivalently, $M' = M$ if $\prec$ is antisymmetric).*

Definition 6 *In a classical manner, if $\prec$ is a preferential relation, we define the preferential-model logic inference $\vdash_\prec$ as follows: $A \vdash_\prec B$ iff each minimal model of A is a model of B.*

Example 1 *To represent the sentences "Generally students are young" and "Lea is a student ", we can use a preferential approach, that is close to the circumscription (McCarthy). An additional predicate "Abnormal" is added and our first sentence is translated to "A student, which is not abnormal is young". Now, if the predicates St, Ab, and Yo, respectively, denotes "Student", "Abnormal" and "Young", then in first order logic, we obtain the following set of formulas:*

$$A \equiv \{St(Lea), \forall x(St(x) \wedge \neg Ab(x)) \rightarrow Yo(x)\}$$

By instantiating the constant Lea to the variable x, we translate the set of formulas into propositional logic and obtain the following set of formulas:

$$A \equiv \{St(Lea), (St(Lea) \wedge \neg Ab(Lea) \rightarrow Yo(Lea)\}$$

Thus :

$$A \equiv \{St(Lea), Ab(Lea) \vee Yo(Lea)\}$$

The set A has an amount of eight interpretations, among them the three following which are the models:

$$\begin{aligned} M_1 &= \{St(Lea), Ab(Lea), Yo(Lea)\} \\ M_2 &= \{St(Lea), \neg Ab(Lea), Yo(Lea)\} \\ M_3 &= \{St(Lea), Ab(Lea), \neg Yo(Lea)\} \end{aligned}$$

In a classical logic, it is impossible to infer from A that Lea is young. Indeed, A has both models were Lea is young (M_1 and M_2), and models were Lea is not young, particularly those where Lea is abnormal (M_3). To obtain the result "Lea is young", we will prefer the models that have fewer abnormal students. Thus, in a preferential model approach, the relation $\prec$ can be defined as: $M \prec M'$ iff "each individual which is abnormal in M is abnormal in M'". Here, we consider that the circumscription is made on the predicate Ab, and all the other predicates vary (not fixed). For instance we can remark that $Yo(Lea)$ varies, it is true in M_1 and false in M_3. The literal $St(Lea)$ could be fixed, but we do not find that necessary. According to this relation, we obtain the following preferences between the models of A: $M_1 \prec M_3$, $M_3 \prec M_1$, $M_2 \prec M_1$, $M_2 \prec M_3$. Therefore A has only one minimal model which is M_2, and Lea is young in this model. We can then infer that Lea is young in this preferential approach. It is then important to infer all the symmetrical of the literal $Yo(Lea)$ with respect to this preference relation.

Symmetry

We recall that a semantic symmetry of A is a permutation σ of the propositional variables of A such that A and $\sigma(A)$ have the same models. Now, we extend the definition of semantic symmetry to preferential-model logics and show how literals could be symmetrical in this non-classic logic but not symmetrical in a classic logic.

Definition 7 (Semantic preferential symmetry) *If $\vdash_\prec$ is a preferential-model inference, A a set of formulas, and σ a permutation defined on the variables of A, then σ is a symmetry of A, iff A and $\sigma(A)$ have the same set of minimal models.*

Definition 8 *Two literals l and $\acute{l}$ are symmetrical in A iff there exists a semantic preferential symmetry σ of A such that $\sigma(l) = \acute{l}$.*

Example 2 *If we take the previous example and add the fact that "John is a student", then we obtain the following formula: A'=$\{St(Lea), St(John), Ab(Lea) \vee Yo(Lea), Ab(John) \vee Yo(John)\}$ which admits nine models. It is easy to see that the individuals Lea and $John$ are symmetrical in both classical and preferential logics in the sense that each predicate literals where Lea appears is symmetrical to the predicate literals where we substitute $John$ to Lea. If we consider the "abnormal" like the pertinent information on which it is based the preference and the permutation σ=$(St(Lea), St(John))(Ab(Lea), Ab(John))$ $(Yo(Lea), Yo(John))$ then we can easily see that σ is a semantic preferential symmetry of the formula A'. Indeed, there is one minimal model that is M=$\{St(Lea), St(John), \neg Ab(Lea), \neg Ab(John),$ $Yo(Lea), Yo(John)\}$ an which is preserved by σ. We can see that the literals $Yo(Lea)$ and $Yo(John)$ are symmetrical as well as $Ab(Lea)$ and $Ab(John)$.*

Now, if we add to A' the information "John is not abnormal", then both $Yo(Lea)$ and $Yo(John)$ as well as $\neg Ab(Lea)$ and $\neg Ab(John)$ remains symmetrical two by two in the preferential logic since M remains the single minimal model of the theory where all the literals are true. However, the literals $Yo(Lea)$ and $Yo(John)$ are not symmetrical in a classical logic. Indeed, the new formula contains the following three extended models:

$$\begin{aligned} M_1 &= \{St(Lea), St(John), Ab(Lea), Yo(Lea), \\ &\quad \neg Ab(John), Yo(John)\} \\ M_2 &= \{St(Lea), St(John), \neg Ab(Lea), Yo(Lea), \\ &\quad \neg Ab(John), Yo(John)\} \\ M_3 &= \{St(Lea), St(John), Ab(Lea), \neg Yo(Lea), \\ &\quad \neg Ab(John), Yo(John)\} \end{aligned}$$

where both literals are not symmetrical. We can see for instance that the model M_3 does not remain a model if we permute $Yo(Lea)$ and $Yo(John)$.

It is then important to see that some literals could be symmetrical in a preferential approach but not symmetrical in a classical logic. This is due to the relaxation of the symmetry conditions in the preference approaches that consider only the minimal models. This information is new and is promising for symmetry reasoning in non classical logics.

We extended the semantic symmetry notion to preferential logics, in the next section, we try to extend the notion of syntactic symmetry to non-classical logics. To do that we chose the X-logic (Siegel, Forget, and Risch 2001) as a baseline framework.

Symmetry in X-Logic

We have seen that it is easy to extend the notion of semantic symmetry to preferential-model logics, but the syntactic symmetry definition in such logics seems to be not trivial. For this purpose, we will use the X-logic (Siegel, Forget, and Risch 2001) formalism that looks to have some syntactic important properties that we will use to extend the notion of syntactic symmetry to non-classical logics.

Definition 9 (X-logic) *Let X be a set of formulas of the propositional logic L (X is not necessary deductively closed). The non-monotonic inference relation $\vdash_X$ is defined by $A \vdash_X B$ iff $\overline{(A \cup B)} \cap X \subseteq \bar{A} \cap X$*

Remark 1
- *In other words, $A \vdash_X B$ if every theorem of $\overline{(A \cup B)}$, which is in X is also a theorem of $\bar{A}$. That is, by adding the knowledge B to A the set of theorems which are in X does not grow. As the classical logic inference $\vdash$ is monotonic, then we also have $\bar{A} \cap X \subseteq \overline{(A \cup B)} \cap X$, and therefore, it is possible to define the X-logic inference $\vdash_X$ by $A \vdash_X B$ iff $\overline{(A \cup B)} \cap X = \bar{A} \cap X$.*
- *For the particular case where $X = F$ (the set of all possible formulas of the logic L), the inference $\vdash_X$ is identical to the classical inference $\vdash$. On other hand if X is empty, then every formula B can be inferred by A.*

The set X can be seen as a potentiometer that regulates the inference. Intuitively, if a formula A encodes an information (a knowledge, or some beliefs..), X can be considered as the set of "pertinent" informations. The set A implies a set B of information, for the X-logic inference, if the addition of B to A does not produce more pertinent formulas than those produced by A alone.

The set X is any set of formula, not necessary closed. It is possible to deduce some properties on the X-logic by adding properties on the set X. For example, if the complement of X, is deductively closed, then the corresponding X-logic will have very interesting properties such as that one of "Cumulativity" (Siegel, Forget, and Risch 2001). If X is the set of positive formulas (formulas that do not contain the logical operators $\rightarrow$ and $\neg$), then we obtain the Closed World Assumption (CWA).

Symmetry

Now we will deal with symmetry in X-logics. We extend the definition of syntactic symmetry to the framework of X-logic and give an extended rule of symmetry that can be used to make short proofs by using symmetrical formulas within this framework.

Definition 10 *Let A be a set of formulas of the propositional logic, X the subset of pertinent formulas on which it is build the X-logic inference $\vdash_X$ and σ is a variable permutation. The permutation σ is a syntactic symmetry of A in the considered X-logic if the following conditions hold.*

1. $\sigma(A) = A$,
2. $\sigma(X) = X$

Now we extend the symmetry rule of Krishnamurthy to the X-logic framework.

Proposition 2 *Let A and B be two formulas or two sets of formulas of the classical logic L and σ a syntactic symmetry of A in the considered X-logic. We have the following rule:*

$$\frac{A \vdash_X B}{A \vdash_X \sigma(B)}$$

***Proof* 1** *To prove that $A \vdash_X \sigma(B)$ we shall prove that $\overline{A \cup \sigma(B)} \cap X = \bar{A} \cap X$. We have by the hypothesis $A \vdash_X B$, that is $\overline{(A \cup B)} \cap X = \bar{A} \cap X$. Since σ is a syntactic symmetry of A in the X-logic, then it preserve the propositional logic theorems. Thus, we have $\sigma(\overline{A \cup B} \cap X) = \sigma(\bar{A} \cap X)$ which is equivalent to $\sigma(\overline{A \cup B}) \cap \sigma(X) = \sigma(\bar{A}) \cap \sigma(X)$. This gives $\overline{\sigma(A \cup B)} \cap \sigma(X) = \overline{\sigma(A)} \cap \sigma(X)$ which is equivalent to $\overline{\sigma(A) \cup \sigma(B)} \cap \sigma(X) = \overline{\sigma(A)} \cap \sigma(X)$. Since both A and X are invariant under σ, then we deduce $\overline{A \cup \sigma(B)} \cap X = \bar{A} \cap \sigma(X)$. Therefore, $A \vdash_X \sigma(B)$.*

Example 3 *Take for instance the student example encoded by the following set of formulas: $A' = \{St(Lea), St(John), Ab(Lea) \vee Yo(Lea), Ab(John) \vee Yo(John)\}$. Consider the set $X = \{\neg Ab(Lea), \neg Ab(John)\}$ and the permutation: $\sigma = (St(Lea), St(John))(Ab(Lea), Ab(John))(Yo(Lea), Yo(John))$. The permutation σ is a X-logic symmetry of A'. We have $A' \vdash_X Yo(Lea)$, and by symmetry we have $A' \vdash_X Yo(John)$ since $\sigma(Yo(Lea)) = Yo(John)$.*

This rule could be used to infer all the symmetrical formulas of an inferred formula in the considered X-logic, its implementation in a theorem prover will help to shorten a proof of a theorem.

In the next section, we extend symmetry to a more general and well known non-monotonic logic that is the Default logic framework introduced by Reiter (Reiter 1980).

Symmetry in Default logic

We investigate in this section the notion of symmetry in default logics and study its relationship with the other non-classical logics discussed before. A Default logic is a non-monotonic logic that is introduced by Reiter (Reiter 1980) to formalize reasoning by default assumptions. Before dealing with symmetry in this logic we summarize some of its background.

Background

Definition 11 *A Default theory T is a pair $\langle D, W\rangle$ where W is a set of logical formulae, called* the background theory, *that formalize the facts that are known for sure. D is a set of* default rules, *each one being of the form:*

$$\frac{Prerequisite : Justification_1, \ldots, Justification_n}{Consequent}$$

Intuitively, this default means that, if the $Prerequisite$ is true, and each $Justification_i$ for all $i \in \{1, \ldots, n\}$ is consistent with our current beliefs, then we are led to believe that the $Consequent$ is true, and infer it (add it to the theory). To define formally the meaning of a default rule, we need to introduce before that, the important notion of extensions in a Default logic.

An extension of a Default theory $T = \langle D, W\rangle$ is a deductively closed set E of formulae that includes W and which verifies: if $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D$ is a default such that $\alpha \in E$, and $\forall i \in \{1, \ldots, n\}, \neg\beta_i \notin E$, then $\gamma \in E$. Formally:

Definition 12 *If $Th(E)$ is the set of logical consequences of the set of formulae E, then E is an extension of the Default theory $T = \langle D, W \rangle$ if and only if $E = \bigcup_{i=0,\ldots,\infty} E_i$ and the following conditions hold: $E_0 = W$, and,*
$\forall i \geq 0, E_{i+1} = Th(E_i) \cup \{\gamma : \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D, \alpha \in E_i, \forall j \in \{1,\ldots,n\}, \neg\beta_j \notin E\}$

A default rule $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma}$ can be applied to a given default theory $T = \langle D, W \rangle$ if its prerequisite α is in W, and the negation of each of its justifications $\neg\beta_j$ is not in the extension E.

Remark 2 *1. When all the defaults of the theory are normal (in the form $\frac{\alpha:\beta}{\beta}$), the condition of the justification $\neg\beta \notin E$ is simplified into $\neg\beta \notin E_i$. We get then a more simple and constructive method to build the extension E.*

2. The default rules may be applied in different order, and this may lead to different extensions E for the same theory T.

3. If a default contains formulae with free variables, it is considered to represent the set of all defaults obtained by giving a value to all these variables.

Example 4 *Take for instance the known default rule "birds typically fly" which is formalized by the following default : $D = \left\{\frac{Bird(X):Flies(X)}{Flies(X)}\right\}$. This rule means that, if X is a bird, and it can be assumed that it flies, then we can conclude that it flies. A background theory containing some facts about birds is the following one: $W = \{Bird(Condor),\ Bird(Penguin),\ \neg Flies(Penguin),\ Flies(Eagle)\}$. We get then the default theory $T = \langle D, W \rangle$.*

According to this default rule, a condor flies because $Bird(Condor)$ is true and the justification $Flies(Condor)$ is not inconsistent with what is currently known. On the contrary we know that a penguin does not fly. Thus, $Bird(Penguin)$ does not allow concluding $Flies(Penguin)$, even if the precondition of the default $Bird(Penguin)$ is true, the justification $Flies(Penguin)$ is inconsistent with what is known. Therefore we get a single extension for the theory T which is:
$E = \{Bird(Condor), Bird(Penguin), \neg Flies(Penguin), Flies(Eagle), Flies(Condor)\}$

The more important notion in a Default logic is the computation of extensions of a Default theory. Based on this set of extensions, researchers defined different semantics for a Default logic. Entailment of a formula from a default theory $T = \langle D, W \rangle$ can be defined in different ways:

Definition 13 *Given a Default theory $T = \langle D, W \rangle$ and the set of all its extensions E_T.*

- **Skeptical approach:** *a formula f is entailed by the default theory T if it is entailed by all its extensions; that is $T \vdash_S f$ iff $\forall E \in E_T, E \vdash f$.*
- **Credulous approach:** *a formula is entailed by a default theory if it is entailed by at least one of its extensions; that is $T \vdash_C f$ iff $\exists E \in E_T : E \vdash f$.*
- **Semi-credulous approach:** *a formula is entailed by a default theory if it is entailed by at least one of its extensions and all of these never entail its negation; that is $T \vdash_{SC} f$ iff $\exists E \in E_T : E \vdash f$ and $\forall E \in E_T, E \nvdash \neg f$.*

Symmetry

Now we introduce the notion of symmetry in Default logic and show how entailment is improved by the property of symmetry. For this logic we distinguish tow levels of symmetry : syntactic symmetry and semantic symmetry. We define in the following both them and study their relationship.

Definition 14 (Semantic symmetry) *Given a Default theory $T = \langle D, W \rangle$, V_T is the set of its variables and E_T the set of all its extensions. A semantic symmetry σ is a variable permutation defined on V_T such that $E_T = E_{\sigma(T)}$.*

In other words, a semantic symmetry of a default theory T is a variable permutation that leaves invariant its set of extensions. It results from this, that each extension $E_i \in E_T$ is transformed by the symmetry σ to another extension $E_j = \sigma(E_i)$. These extensions are what we call symmetrical extensions. It is then possible to get family of symmetrical extensions without duplication of efforts if we know that E_i is an extension and we have at hand the symmetry group of the theory T. Unfortunately, computing semantic symmetry is time consuming, since it needs to compute all the extensions. In the following, we will define the syntactic symmetry, and show that it is a sufficient condition for semantic symmetry which can be computed efficiently.

Definition 15 (Syntactic symmetry) *Given a Default theory $T = \langle D, W \rangle$, a syntactic symmetry is a permutation σ defined on the set V_T of variables of T that lives the theory T invariant. That is, $\sigma(T) = T$, more precisely, the following conditions hold: $\sigma(D) = D$ and $\sigma(W) = W$.*

Example 5 *Take for instance the student example discussed before and consider the following default theory $T = (\{\frac{St(X):\neg Ab(X)}{Yo(X)}\}, \{St(Lea), St(John)\})$. The permutation σ=$(St(Lea), St(John))(Ab(Lea), Ab(John))$ $(Yo(Lea), Yo(John))$ is a syntactic symmetry since it leaves T invariant. By considering all the terminal instantiations of the free variable of the defaults of the theory T we can see that T has one extension $E = \{St(Lea), St(John), Yo(Lea), Yo(John)\}$ which remains invariant under σ. We conclude that σ is also a semantic symmetry.*

Now, if we add $Yo(John)$ to the facts of the theory T of the previous example, then we get the theory a new theory T' for which σ is not a syntactic symmetry. However, σ remains a semantic symmetry of T' since the new theory T' has the same extension E as T. This, illustrate that semantic symmetry includes syntactic symmetry.

We give in the following theorem the relationship between semantic symmetry and syntactic symmetry of a Default logic.

Theorem 1 *Given a Default theory T, if σ is a syntactic symmetry of T, then σ is a semantic symmetry of T.*

Proof **2** *The proof is trivial. Since σ is a syntactic symmetry, then $\sigma(T) = T$. Thus, $E_T = E_{\sigma(T)}$.*

Now we can introduce the new symmetry inference rule in Default logics. Take for instance the skeptical inference $\vdash_S$.

Proposition 3 *Let T be a default theory, f a formula and σ a syntactic symmetry of T. We have the following rule:*

$$\frac{T \vdash_S f}{T \vdash_S \sigma(f)}$$

***Proof* 3** *To show that $T \vdash_S \sigma(f)$ requires to prove that $\forall E_k \in E_T, E_k \vdash \sigma(f)$. Let $E_i \in E_T$, by the definition of the symmetry σ there exists E_j such that $E_i = \sigma(E_j)$. By the hypothesis, we have $T \vdash_S f$. Thus $E_j \vdash f$, and then $\sigma(E_j) \vdash \sigma(f)$. Therefore $E_i \vdash \sigma(f)$. We conclude that $T \vdash_S \sigma(f)$.*

Remark 3 *The previous rule is also valid for both the credulous ($\vdash_C$) and the semi-credulous ($\vdash_{SC}$) inferences.*

Example 6 *Take Example 5. In the theory T, we have $T \vdash_S Yo(Lea)$. Then by the symmetry inference rule we can make the entailment $T \vdash_S \sigma(Yo(Lea))$. Thus, $T \vdash_S Yo(John)$.*

Now we give an important proposition that uses symmetry to compute the set of extensions.

Proposition 4 *Given a Default theory $T = \langle D, W\rangle$, a subset of formulae E, and a syntactic symmetry σ of T, then E is an extension of T iff $\sigma(E)$ is an extension.*

***Proof* 4** *Suppose that E is an extension. The permutation σ is a syntactic symmetry, then by Theorem 1 it is a semantic symmetry. It then preserves the set E_T of extensions of T. It results from this, that $\sigma(E) \in E_T$. The converse can be shown in the same way by considering the inverse symmetry σ^{-1} of σ.*

Now we discuss the relationship of symmetrical extensions of a Default theory and their corresponding subsets of defaults that are used to build them.

Proposition 5 *Given a Default theory $T = \langle D, W\rangle$, a subset $D_1 \subset D$, and a syntactic symmetry σ, then there exists an extension E^{D_1} of T obtained by application of the defaults of D_1, if and only if, there exists an extension $E^{\sigma(D_1)}$ of T obtained by application of the defaults of $\sigma(D_1)$.*

***Proof* 5** *Suppose that E^{D_1} is an extension of $T = \langle D, W\rangle$ obtained by application of the defaults of D_1. We shall prove that there exists an extension $E^{\sigma(D_1)}$ of T that can be obtained by application of the defaults of $\sigma(D_1)$. Since E^{D_1} is an extension of T, then from Definition 12, we have the following: $E^{D_1} = \bigcup_{i=0,\ldots,\infty} E_i$ where: $E_0 = W$, and, $\forall i \geq 0, E_{i+1} = Th(E_i) \cup \{\gamma : \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D_1, \alpha \in E_i, \forall j \in \{1,\ldots,n\}, \neg\beta_j \notin E^{D_1}\}$. To show the existence of the extension $E^{\sigma(D_1)}$ of T, we set $E' = \sigma(E^{D_1})$, $E'_0 = W$, $\forall i \geq 0$, $E'_{i+1} = Th(E'_i) \cup \{\sigma(\gamma) : \frac{\sigma(\alpha):\sigma(\beta_1),\ldots,\sigma(\beta_n)}{\sigma(\gamma)} \in \sigma(D_1), \sigma(\alpha) \in E'_i, \forall j \in \{1,\ldots,n\}, \neg\sigma(\beta_j) \notin E'\}$ and prove that E' is an extension of T and $E' = E^{\sigma(D_1)}$. To show that E' is an extension of T, we have to show that $E' = \bigcup_{i=0,\ldots,\infty} E'_i$. To do that, first we have to show that $\forall i \geq 0$, $E'_i = \sigma(E_i)$. We prove this property by induction on i. For the first step ($i = 0$), we have $E'_0 = W$, thus $E'_0 = \sigma(W)$ since σ is a syntactic symmetry of T. It results that $E'_0 = \sigma(E_0)$. Now, we suppose that the property is verified at the step i, that is, $E'_i = \sigma(E_i)$, we shall show that the property holds at the step $i + 1$, that is, $E'_{i+1} = \sigma(E_{i+1})$. By the definition of E'_{i+1}, we have $E'_{i+1} = Th(E'_i) \cup \{\sigma(\gamma) : \frac{\sigma(\alpha):\sigma(\beta_1),\ldots,\sigma(\beta_n)}{\sigma(\gamma)} \in \sigma(D_1), \sigma(\alpha) \in E'_i, \forall j \in \{1,\ldots,n\}, \neg\sigma(\beta_j) \notin E'\}$. By the hypothesis of the induction and the definition of E', we can rewrite E'_{i+1} as: $E'_{i+1} = Th(\sigma(E_i)) \cup \{\sigma(\gamma) : \frac{\sigma(\alpha):\sigma(\beta_1),\ldots,\sigma(\beta_n)}{\sigma(\gamma)} \in \sigma(D_1), \sigma(\alpha) \in \sigma(E_i), \forall j \in \{1,\ldots,n\}, \neg\sigma(\beta_j) \notin \sigma(E^{D_1})\}$. On other hand we have $\sigma(Th(E_i)) = Th(\sigma(E_i))$ since the symmetry σ verifies the following: if $A \vdash B$, then $\sigma(A) \vdash \sigma(B)$. Therefore $E'_{i+1} = \sigma(Th(E_i)) \cup \sigma(\{\gamma : \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D_1, \alpha \in E_i, \forall j \in \{1,\ldots,n\}, \neg\beta_j \notin E^{D_1}\})$. We deduce that $E'_{i+1} = \sigma(Th(E_i) \cup \{\gamma : \frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma} \in D_1, \alpha \in E_i, \forall j \in \{1,\ldots,n\}, \neg\beta_j \notin E^{D_1}\})$. This implies that $E'_{i+1} = \sigma(E_{i+1})$, and then we prove the property. Now we shall prove that E' is an extension of T. By definition $E' = \sigma(E^{D_1})$, since E^{D_1} is by the hypothesis an extension of T ($E^{D_1} = \bigcup_{i=0,\ldots,\infty} E_i$), then $E' = \sigma(\bigcup_{i=0,\ldots,\infty} E_i)$, thus $E' = \bigcup_{i=0,\ldots,\infty} \sigma(E_i)$. On other hand we have shown that $\forall i \geq 0$, $E'_i = \sigma(E_i)$, thus $E' = \bigcup_{i=0,\ldots,\infty} E'_i$, it results that E' is an extension of T. As E' is an extension of T obtained by application of the defaults of $\sigma(D_1)$, then we obtain $E' = E^{\sigma(D_1)}$. Finally, we conclude that there exists an extension $E^{\sigma(D_1)}$ of T that can be obtained by application of the defaults of $\sigma(D_1)$. The converse can be shown in the same way by considering the inverse symmetry σ^{-1} of σ.*

Now we will show that if the application of a default leads to an extension of the considered theory, then the application of each of all its symmetrical defaults leads to an extension of the theory too.

Proposition 6 *If $T = \langle D, W\rangle$ is a Default theory and σ one of its syntactic symmetries, then there exists an extension of T where a default d is applied, if an only if, there exists an extension where its symmetrical default $\sigma(d)$ is applied.*

***Proof* 6** *Let E be an extension of T where d is applied. This implies that there exists a subset $D_1 \subset D$ such that $d \in D_1$ and $E = E^{D_1}$. By Proposition 5, we deduce that $E^{\sigma(D_1)}$ is an extension of T where $\sigma(d) \in \sigma(D_1)$ is applied.*

From the previous proposition we deduce the following property:

Corollary 1 *Given a Default theory $T = \langle D, W\rangle$, a default $d \in D$, and a syntactic symmetry σ of T, then there does not exist an extension of T where the default d is applied, if an only if, there does not exist an extension where its symmetrical default $\sigma(d)$ is applied.*

In Default logic we have the advantage that both syntactic and semantic symmetry are defined. The Default logic is augmented with a symmetry inference rule that can be used to shorten proofs. Besides, in the Default logic symmetry can be used to compute the set of extensions. Indeed, when an extension is identified during the enumeration, we can deduce all of its symmetrical extensions without additional efforts (Proposition 4). In other words the isomorphic branches in the search tree corresponding to the application of symmetrical default sub-sets lead to symmetrical extensions, then we need to explore only one branch and prune the search space corresponding to the others. In case of failure to get an extension when applying a default d, Corollary 1 allows us to prune all the research subspaces corresponding to its symmetrical defaults $\sigma(d)$

Conclusion and perspectives

The main purpose behind this work is to extend the notion of symmetry to non-classical logics formalisms. We defined the notion of semantic symmetry in preferential-model logics and showed how some information can be inferred by using this symmetry extension in preferential logics while such deductions could not be done in a classical logic. The other point studied here is the extension of syntactic symmetry definition to the framework of X-logics where a new symmetry inference rule was given. Finally we defined both semantic and syntactic symmetries in the more general framework of Default logic. We showed how symmetry can be used to improve the search of extensions and introduced a new symmetry inference that can be used to make short proofs. In a future work, one can try to study in depth the relationship between the semantic symmetry in preferential logic and the syntactic symmetry defined in the X-logics and establish their relationship with the symmetry definitions introduced in Default logics. An other point that we want to investigate is to implement symmetry to speed up enumeration algorithms that are used to compute extensions of default logics and provide a proof procedure that will take advantage of the new symmetry rule. Finally, we are looking to study the relationship of Default logics and Answer Set Programming (ASP) in order to compute symmetrical extensions by mean of symmetrical Answer Sets and Stable Models.

References

Aloul, F. A.; A.Ramani; Markov, I. L.; and Sakallak, K. A. 2003. Solving difficult sat instances in the presence of symmetry. *In IEEE Transaction on CAD, vol. 22(9)* 1117–1137.

Aloul, F. A.; Ramani, A.; Markov, I. L.; and Sakallak, K. A. 2004. Symmetry breaking for pseudo-boolean satisfiabilty. *In ASPDAC'04* 884–887.

Backofen, R., and Will, S. 1999. Excluding symmetries in constraint-based search. In *Principle and Practice of Constraint Programming - CP'99*.

Benhamou, B., and Sais, L. 1992. Theoretical study of symmetries in propositional calculus and application. *Eleventh International Conference on Automated Deduction, Saratoga Springs,NY, USA*.

Benhamou, B., and Sais, L. 1994. Tractability through symmetries in propositional calculus. *Journal of Automated Reasoning (JAR),* 12:89–102.

Benhamou, B.; Sais, L.; and Siegel, P. 1994. Two proof procedures for a cardinality based language. *in proceedings of STACS'94, Caen France* 71–82.

Benhamou, B. 1994. Study of symmetry in constraint satisfaction problems. *In Proceedings of the 2nd International workshop on Principles and Practice of Constraint Programming - PPCP'94*.

Besnard, P., and Siegel, P. 1988. The preferential-models approach in nonmonotonic logics - in non-standard logic for automated reasoning. In *Academic Press*, 137–156. ed. P. Smets.

Bossu, G., and Siegel, P. 1982. Nonmonotonic reasoning and databases. In *Advances in Data Base Theory*, 239–284.

Bossu, G., and Siegel, P. 1985. Saturation, nonmonotonic reasoning and the closed-world assumption. *Artif. Intell.* 25(1):13–63.

Crawford, J.; Ginsberg, M. L.; Luck, E.; and Roy, A. 1996. Symmetry-breaking predicates for search problems. In *KR'96: Principles of Knowledge Representation and Reasoning*. San Francisco, California: Morgan Kaufmann. 148–159.

Fahle, T.; Schamberger, S.; and Sellmann, M. 2001. Symmetry breaking. In *International conference on constraint programming*, volume 2239 of *LNCS*, 93–108. Springer Verlag.

Focacci, F., and Milano, M. 2001. Global cut framework for removing symmetries. In *International conference on constraint programming*, volume 2239 of *LNCS*, 77–82. Springer Verlag.

Freuder, E. 1991. Eliminating interchangeable values in constraints satisfaction problems. *Proc AAAI-91* 227–233.

Gent, I. P., and Smith, B. M. 2000. Symmetry breaking during search in constraint programming. In *Proceedings ECAI'2000.*

Gent, I. P.; Hervey, W.; Kesley, T.; and Linton, S. 2003. Generic sbdd using computational group theory. In *Proceedings CP'2003*.

Gent, I.; Harvey, W.; and Kelsey, T. 2002. Groups and constraints: Symmetry breaking during search. In *International conference on constraint programming*, volume 2470 of *LNCS*, 415–430. Springer Verlag.

Kraus, S.; Lehmann, D. J.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence* 44(1-2):167–207.

Krishnamurty, B. 1985. Short proofs for tricky formulas. *Acta informatica* (22):253–275.

McCarthy, J. Circumscription-a form of non-monotonic reasoning. *Artificial Intelligence* 13.

Puget, J. F. 1993. On the satisfiability of symmetrical constrained satisfaction problems. In *In J. Kamorowski and Z. W. Ras,editors, Proceedings of ISMIS'93, LNAI 689*.

Puget, J. 2002. Symmetry breaking revisited. In *International conference on constraint programming*, volume 2470 of *LNCS*, 446–461. Springer Verlag.

Reiter, R. 1980. A logic for default reasoning. *Artificial Intelligence* 81–132.

Roney-Dougal, C. M.; Gent, I. P.; Kelsey, T.; and Linton, S. A. 2004. Tractable symmetry breaking using restricted search trees. *In, proceedings of ECAI'04* 211–215.

Shoam, Y. 1987. A semantical approach to nonmonotonic logic. In *IJCAI*, 388–392.

Siegel, P.; Forget, L.; and Risch, V. 2001. Preferential logics are x-logics. *Journal of Logic and Computation* 11(1):71–83.

Szeider, S. 2005. The complexity of resolution with generalized symmetry rules. *Theory of Computing Systems* 38(2):171–188.

Walsh, T. 2006. General symmetry breaking constraints. *In, proceedings of CP'06* 650–664.