Nonmonotonic Features and Uncertainty in Reasoning with Analogical Proportions

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Abstract

The paper first reconsiders one of the main patterns of plausible reasoning proposed by Polya, namely "a and b are analogous, a is true, then b true is more credible", from a nonmonotonic reasoning point of view. A representation of the statement "a and b are analogous" is proposed in the setting of the preferential entailment system for nonmonotonic consequences relations. This representation is then related to the logical definition recently proposed for analogical proportions, i.e. for statements of the form "a is to b as c is to d". Then the last part of the paper introduces a possibilistic logic treatment of uncertainty associated with observed analogical proportions for extrapolation reasoning. Namely, when an analogical proportion holds for some observable features describing cases a, b, c, and d, we may assume that such a proportion still holds between the other features of d, and the corresponding, observable and known features of a, b, and c, even when the data are pervaded with uncertainty.

Analogical reasoning and nonmonotonic reasoning have in common to lead to defeasible plausible conclusions. Still they have remained almost unrelated up to a few exceptions. While considerable and successful efforts have been developed in the last thirty years for formalizing nonmonotonic reasoning, less attempts have been made for providing a logical modeling of analogical reasoning, in spite of the considerable interest for the use of analogy as a powerful heuristics (Gentner 1989; Sowa and Majumdar 2003). Moreover the existing proposals mainly use first order logic frameworks (Davies and Russell 1987; Melis and Veloso 1998), or even second order settings (Schmid et al. 2003), while in the case of analogical proportions, i.e. statements of the form "a is to b as c is to d", a propositional logic representation has been recently proposed (Miclet and Prade 2009).

In this paper, we start from one of the patterns of plausible reasoning proposed by (Polya 1954) about half a century ago. This pattern "if a and b are analogous, and a is true, then b true is more credible", refers explicitly to the idea of analogy. In this paper, we propose a modeling of "a and b are analogous" in terms of a nonmonotonic consequence relation, and we then provide a counterpart of the inference pattern in the setting of the preferential entailment system (Kraus, Lehmann, and Magidor 1990). We then provide a short background on the propositional logic modeling of analogical proportions, and establish some agreement relations between analogical proportions and our formalization of the relation being "analogous". In the last main part of the paper we extend an inference mechanism based on analogical proportions in order to handle uncertain proportions, or uncertain facts, in the setting of possibilistic logic (already proved to be useful for encoding default reasoning).

Polya's analogy-based pattern of plausible reasoning revisited

Polya (1945) in his famous book "How to Solve It?" advocates the idea that analogical reasoning plays an important role in mathematical problem solving. In particular, he recommends the reader when faced to a problem to ask herself (or himself) questions such as "Can you find a problem analogous to your problem and solve that?" Later, (Polya 1954) proposed patterns of plausible reasoning in order to provide a more accurate view of reasoning mechanisms at work in problem solving. One of these patterns reads

a and b are analogous
a is true
———————————
b true is more credible

For (Polya 1954), this pattern is obtained from the more basic pattern

h implies a
a is true
———————————
h true is more credible

by combining it with the deductive pattern

$$\{h \text{ true more credible}, h \text{ implies } b\} \vdash \{b \text{ true more credible}\}$$

and "summarizing" "h implies a" and "h implies b" by "a and b are analogous".

In the following we propose another view of "being analogous". Informally speaking, the idea is that a and b are analogous if b is true in the normal situations where a is true, and conversely a is true in the normal situations where b is true. This amounts to say that when a is true b generally follows and vice-versa. So we have to model an exception-tolerant entailment. For that purpose, we use a nonmonotonic consequence relation $\mid\!\sim$ obeying the postulates of preferential entailment system P (for short) (Kraus, Lehmann, and Magidor 1990), that we briefly recall below. Indeed, system P has been shown to be at the root of nonmonotonic reasoning, being equivalent to several other non monotonic reasoning systems based on different views of modeling conditionals (Benferhat, Dubois, and Prade 1997).

System P is characterized by a reflexive nonmonotonic consequence relation $\mid\!\sim$ obeying the following postulates:

- Left Logical Equivalence: $\models p \equiv q$ and $p \mid\!\sim r$ imply $q \mid\!\sim r$
- Right Weakening: $p \mid\!\sim q$ and $q \models r$ imply $p \mid\!\sim r$
- Cautious Monotony: $p \mid\!\sim q$ and $p \mid\!\sim r$ imply $p \wedge q \mid\!\sim r$
- Cut: $p \mid\!\sim q$ and $p \wedge q \mid\!\sim r$ imply $p \mid\!\sim r$
- OR: $p \mid\!\sim r$ and $q \mid\!\sim r$ imply $p \vee q \mid\!\sim r$

and some of its noticeable consequences are:

- AND: $p \mid\!\sim q$ and $p \mid\!\sim r$ imply $p \mid\!\sim q \wedge r$
- S: $p \wedge q \mid\!\sim r$ implies $p \mid\!\sim q \rightarrow r$
- Consequent Modus Ponens: $p \mid\!\sim q \rightarrow r$ and $p \mid\!\sim q$ imply $p \mid\!\sim r$

Using a nonmonotonic consequence relation $\mid\!\sim$ obeying the postulates of system P, the above-mentionned view of "being analogous" amounts to state the following condition for having a and b analogous, which is denoted $a \approx b$:

$$a \approx b \text{ iff } a \mid\!\sim b \text{ and } b \mid\!\sim a \qquad (1)$$

The intuition underlying (1) amounts to say that a and b are analogous as soon as they are generally true simultaneously. This definition has some consequences which may look troublesome at a first glance. Indeed according to (1), $a \mid\!\sim b$ and $a \mid\!\sim c$ entail $a \wedge b \approx a \wedge c$ (this can be seen as follows: applying cautious monotony, we get $a \wedge b \mid\!\sim c$, and then applying AND to this formula together with $a \wedge b \mid\!\sim a$ obtained by reflexivity and right weakening, we get $a \wedge b \mid\!\sim a \wedge c$; the converse is proved similarly). Thus if we admit the two default rules "birds fly" and "bird are omnivorous", then we can conclude that "flying birds" are analogous to "omnivorous birds". Although it may sound strange, it only amounts to mean that the two kinds of bird more or less coincide, but certainly not that "omnivorous" and "flying" are synonymous.

Using the inference rules of System P, definition (1) can also be rewritten under another form. Since by reflexivity $a \mid\!\sim a$, by AND property we get $a \mid\!\sim a \wedge b$. Similarly, $b \mid\!\sim a \wedge b$, and by OR rule, we obtain

$$\text{if } a \approx b \text{ then } a \vee b \mid\!\sim a \wedge b$$

By Right Weakening and by AND, $a \vee b \mid\!\sim a \wedge b$ is equivalent to $a \vee b \mid\!\sim a$ and $a \vee b \mid\!\sim b$. By applying the Cautious Monotony rule above letting $p = a \vee b$, $q = a$ and $r = b$, we obtain $a \mid\!\sim b$, and similarly $b \mid\!\sim a$. This shows that $a \vee b \mid\!\sim a \wedge b$ is equivalent to $a \mid\!\sim b$ and $b \mid\!\sim a$, and thus,

$$a \approx b \text{ iff } a \vee b \mid\!\sim a \wedge b \qquad (2)$$

This expresses that the normal models of $a \vee b$ are in the intersection of the sets of models of a and of b.

Note that one may also think of a definition still stronger than (1), namely $a \wedge \neg b \mid\!\sim \bot$ and $\neg a \wedge b \mid\!\sim \bot$ (indeed by S property, it entails (1)), which expresses that there is no model of $a \wedge \neg b$ or of $\neg a \wedge b$. This is much too strong, since it would amount to state that $a \vdash b$ and $b \vdash a$.

However, a definition of "being analogous" weaker than (1) seems more interesting. By property S, $a \vee b \mid\!\sim a \wedge b$ entails $\mid\!\sim (a \vee b) \rightarrow (a \wedge b)$, which is equivalent (by double application of Right Weakening) to $\mid\!\sim (\neg a \wedge \neg b) \vee (a \wedge b)$, still equivalent to $\mid\!\sim (\neg a \vee b) \wedge (a \vee \neg b)$, i.e., $\mid\!\sim (a \rightarrow b) \wedge (b \rightarrow a)$. This is also equivalent to $\mid\!\sim (a \rightarrow b)$ and $\mid\!\sim (b \rightarrow a)$ (by Right Weakening and by AND). But, this does not entail (1) [1]. Thus, it leads to the weaker definition of "a being analogous b", which is denoted $a \sim b$:

$$a \sim b \text{ iff } \mid\!\sim (a \rightarrow b) \wedge (b \rightarrow a) \qquad (3)$$

and

$$a \approx b \text{ implies } a \sim b \qquad (4)$$

Thus $a \sim b$ should be read as "a is weakly analogous to b". Then using Consequent Modus Ponens, it can be easily checked that both

$$a \sim b \text{ and } \mid\!\sim a \text{ imply } \mid\!\sim b$$

$$a \approx b \text{ and } \mid\!\sim a \text{ imply } \mid\!\sim b$$

which correspond to Polya's pattern. It is clear that these two options offer relaxed version of a classical logic modeling, namely,

$$\vdash a \equiv b \text{ and } \vdash a \text{ imply } \vdash b$$

[1] To see that $\mid\!\sim (a \rightarrow b)$ is weaker than $a \mid\!\sim b$, it is convenient to use the possibility theory semantics of System P (Benferhat, Dubois, and Prade 1997). Then the latter expression is equivalent to $\Pi(a \wedge b) > \Pi(a \wedge \neg b)$, while the other reads $\Pi(\neg a \vee b) > \Pi(a \wedge \neg b)$, where Π is a (max-decomposable) possibility measure (w.r.t. disjunction), and thus $\Pi(\neg a \vee b) = max(\Pi(\neg a), \Pi(a \wedge b))$. Thus, $a \vee b \mid\!\sim a \wedge b$ translates into $\Pi(a \wedge b) > \Pi((a \vee b) \wedge \neg(a \wedge b))$, while $\mid\!\sim (a \vee b) \rightarrow (a \wedge b)$ is semantically equivalent to $max(\Pi(\neg a \wedge \neg b), \Pi(a \wedge b)) > \Pi((a \vee b) \wedge \neg(a \wedge b))$, which is a weaker constraint.

which would reduce "being analogous" to "being semantically identical".

Background on the analogical proportion

Before relating the nonmonotonic view of an analogy relation that we have just proposed to the notion of analogical proportion, let us briefly review the set and logical interpretations of an analogical proportion. We first introduce the set interpretation of analogical proportion since it provides a view which is probably easier to grasp. Moreover, this is a suitable viewpoint when it comes to applications where data representations are more sophisticated than atomic Boolean values. In that context, the simplest way to understand the meaning of a statement such as "a is to b as c is to d" is to consider the 4 items a, b, c, d as subsets of binary features (belonging to a referential X) possessed by these items. As usual, $\overline{a} = X \setminus a$ denotes the complement of a, i.e. the features in X not in a. As suggested in (Lepage 2001), (Stroppa and Yvon 2005), a, b, c, d are in analogical proportion if a can be changed into b and c into d by "adding and deleting the same elements". Using a formal counterpart of this idea (Miclet and Prade 2009), the analogical proportion, denoted $a : b :: c : d$, is specified by the following pair of constraints:

$$a \cap \overline{b} = c \cap \overline{d} \text{ and } \overline{a} \cap b = \overline{c} \cap d \quad (A_{set})$$

It is straighforward to check that such a formal definition satisfies the properties that are usually assumed for an analogical proportion, namely:

- $a : b :: a : b$ and $a : a :: b : b$ hold, but $a : b :: b : a$ does not hold in general;
- if $a : b :: c : d$ holds then $a : c :: b : d$ should hold (central permutation);
- if $a : b :: c : d$ holds then $c : d :: a : b$ should hold (symmetry).

As we can see, an analogical proportion $a : b :: c : d$ is interpreted as an equality between two pairs $(a \cap \overline{b}, \overline{a} \cap b) = (c \cap \overline{d}, \overline{c} \cap d)$. The second property allows us permuting the means in a proportion and by combining with the third property, it is an easy game to get the result below:

Property 1 *An analogical proportion $a : b :: c : d$ has 8 equivalent forms:*

(i) If $a : b :: c : d$ holds then $a : c :: b : d$ holds (by central permutation)

(ii) If $a : b :: c : d$ holds then $d : b :: c : a$ holds (by symmetry + central permutation + symmetry)

(iii) If $a : b :: c : d$ holds then $d : c :: b : a$ holds (by ii + central permutation)

(iv) If $a : b :: c : d$ holds then $c : d :: a : b$ holds (by symmetry)

(v) If $a : b :: c : d$ holds then $b : d :: a : c$ holds (by i + symmetry)

(vi) If $a : b :: c : d$ holds then $c : a :: d : b$ holds (by iv + central permutation)

(vii) If $a : b :: c : d$ holds then $b : a :: d : c$ holds (by v + central permutation)

Table 1: Analogy classes

a	**b**	**c**	**d**	**b**	**a**	**c**	**d**	**c**	**b**	**a**	**d**
a	c	b	d	b	c	a	d	c	a	b	d
d	b	c	a	d	a	c	b	d	b	a	c
d	c	b	a	d	c	a	b	d	a	b	c
c	d	a	b	c	d	b	a	a	d	c	b
b	d	a	c	a	d	b	c	b	d	c	a
c	a	d	b	c	b	d	a	a	c	d	b
b	a	d	c	a	b	d	c	b	c	d	a

This property tells us that if $a : b :: c : d$ holds then a total of 8 permutations (among 24) hold, including this one, and all the seven ones appearing in the above Property. This constitutes the first column of Table 1. As noticed in (Lepage 2001), the 16 remaining permutations are divided into two other classes of equivalent analogical proportions, corresponding to columns 2 and 3 of Table 1 which exhibits the three classes with representative on top in bold font.

Leaving the set interpretation for the remaining of the paper, we now interpret analogical proportion over the Boolean lattice $\mathbb{B} = \{0, 1\}$ with the standard operators $\vee, \wedge, \neg$. Following the lines of (Miclet and Prade 2009), we use a direct translation of the set theoretic definitions where $\cup$ is replaced with $\vee$, $\cap$ with $\wedge$, complementarity with $\neg$. To simplify the notation, we write $u \equiv v$ to denote the formula $u \to v \wedge v \to u$. Of course, this logical notation underlies the equality of the truth values for u and v. This leads to

Definition 1 *$a : b :: c : d$ holds iff*

$((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$ *is true* (A_{Bool})

Thus, $a : b :: c : d$ is now viewed as a Boolean operator with truth value 1 only for the 6 following 4-tuples (among 16 cases):

0	0	0	0
1	1	1	1
0	0	1	1
1	1	0	0
0	1	0	1
1	0	1	0

For all the other values of (a, b, c, d), $a : b :: c : d$ is false. This relation over $\mathbb{B}^4$ satisfies the properties of analogical proportion and several equivalent writings have been proposed in (Miclet and Prade 2009) but the most useful in our context is the following one (clearly equivalent to Definition 1):

Property 2 *$(a : b :: c : d)$ holds iff*

$$((a \to b) \equiv (c \to d)) \wedge ((b \to a) \equiv (d \to c)) \textit{is true}$$

Note that the above expression parallels the difference-based view of the analogical proportion expressed by (A_{set}). The results holding for the set theoretic interpretation are easy to restate in the logical setting (see (Prade and Richard 2010a) for instance). For instance, the proposition below investigates the behavior of analogical proportion with respect to the negation operator:

Property 3 *If $a : b :: c : d$ holds then $\neg b : \neg a :: c : d$ holds*
If $a : b :: c : d$ holds then $\neg a : d :: \neg b : c$ holds
If $a : b :: c : d$ holds then $\neg a : \neg b :: d : c$ holds
If $a : b :: c : d$ holds then $\neg a : \neg b :: \neg c : \neg d$ holds
$a : b :: \neg b : \neg a$ holds.

It appears that we have to be careful with "common sense" reasoning where we could falsely consider that $a : b :: \neg a : \neg b$ holds, while it does not hold in general: for instance, $1 : 0 :: 1 : 0$ holds, but $1 : 0 :: 0 : 1$ does not hold.

Agreement between analogical proportions and the idea of being analogous

Thus, the logical expression of an analogical expression is $a : b :: c : d = (a \to b \equiv c \to d) \wedge (b \to a \equiv d \to c)$. This section aims at showing some agreement between this definition and the weak view of "being analogous". Indeed, using the weak definition of "being analogous" (i.e., expression 3), we can check that the following pattern, which fits the intuition, holds

$$\frac{\begin{array}{l}\mid\!\sim a : b :: c : d\\ a \sim b\end{array}}{c \sim d}$$

Indeed $\mid\!\sim a : b : c : d$ entails $\mid\!\sim (a \to b) \equiv (c \to d)$. Since $a \sim b$ entails $\mid\!\sim a \to b$, by Right Weakening and Consequent modus ponens, we get $\mid\!\sim c \to d$. Similarly, one can prove $\mid\!\sim d \to c$, which by AND property gives the result. In fact, $\models a : b :: c : d$ and $c \sim d$ imply $a \sim b$, since due to Right Weakening, $\models p$ implies $\mid\!\sim p$.

Conversely, it can be checked that two pairs of analogous items make a *defeasible* analogical proportion:

$$\frac{a \sim b, c \sim d}{\mid\!\sim a : b :: c : d}$$

Indeed $a \sim b$ and $c \sim d$ entail $\mid\!\sim a \to b$ and $\mid\!\sim c \to d$, and thus $\mid\!\sim (a \to b) \wedge (c \to d)$, and by Right Weakening $\mid\!\sim (a \to b) \to (c \to d)$. Similarly, they entail $\mid\!\sim (c \to d) \to (a \to b)$, and thus $\mid\!\sim (a \to b) \equiv (c \to d)$. For the same reason, $a \sim b$ and $c \sim d$ entail $\mid\!\sim b \to a$ and $\mid\!\sim d \to c$, thus $\mid\!\sim (b \to a) \equiv (d \to c)$, and finally $\mid\!\sim a : b :: c : d$. This property highlights again the fact that analogical proportions between 4 items is a kind of "higher order" relation comparing the link between a and b with the link between c and d. Indeed, as can be seen in the truth table of the analogical proportion, when a and b are true simultaneously, the fact that the analogical proportion $a : b :: c : d$ holds does not require that c and d to be true: indeed they may also be simultaneously false.

Since $\mid\!\sim a : b :: c : d$ is equivalent to $\mid\!\sim a : c :: b : d$ (by Property 1), we obtain the following pattern, by combining the two previous patterns

$$\frac{a \sim b, c \sim d, a \sim c}{b \sim d}$$

This pattern is reminiscent of the "geometrical" view of the analogical proportion, as a parallelogram a, b, c, d with parallel and equal sides (see, e.g. (Prade and Richard 2010a)).

Lastly, another way of connecting Polya's pattern with analogical proportions would be to interpret an analogical proportion $a : b :: c : d$ as a statement of the form $(a : b) \approx (c : d)$ where $(a : b)$ is considered as a complex statement. Indeed, with this reading it is clear that from $(a : b) \mid\!\sim (c : d)$ and the observation that $(a : b)$ holds, we non monotonically infer that $(c : d)$. Besides, a statement such as $(c : d)$ could be also understood as $c \sim d$. Then from $(c : d)$ and c, we could non monotonically infer d, since $c \sim d$ is nothing but $\mid\!\sim (c \to d) \wedge (d \to c)$. In other words, with this view, from $a : b :: c : d$ and the facts $(a : b), c$, we infer d by a two-stepped non monotonic inference. At the beginning of the following section, we are going to encounter a very similar pattern.

Reasoning with analogical proportions in presence of uncertainty

In its simplest form, analogical reasoning, without any reference to the notion of proportion, is usually viewed as a way to infer some new fact on the basis of one observation. Analogical reasoning has been mainly formalized in the setting of first order logic (Davies and Russell 1987; Melis and Veloso 1998). A basic pattern for analogical reasoning is then to consider 2 terms s and t, to observe that they share a property P, and knowing that another property Q also holds for s, to infer that it holds for t as well (under the implicit hypothesis that P determines Q inasmuch as $\nexists u\ P(u) \wedge \neg Q(u)$). This is known as the "analogical jump" and can be described with the following simplified inference pattern:

$$\frac{P(s)\ \ P(t)\ \ Q(s)}{Q(t)}$$

This pattern, quite different from Polya's pattern, may be directly related to the idea of analogical proportion in different ways. First, taking advantage that "P(s) is to P(t) as Q(s) is to Q(t)" (indeed they are similar changing s into t), the above pattern may be restated as

$$\frac{\begin{array}{l}\text{P(s) : P(t) :: Q(s) : Q(t)}\\ \text{P(s), P(t), Q(s)}\end{array}}{\text{Q(t)}}$$

which is a valid pattern of inference, as we shall see in the next subsection. Each piece of information may also be encoded in a binary way according to the presence or the absence of P, Q, s, or t in the piece of information, as in the table below, and the encoding d of $Q(t)$ can be uniquely determined by completing the values of a, b, and c for the different features according to the 4-tuples of values that make $a : b :: c : d$ true (see the table after Definition 1):

	P	Q	s	t	
a	1	0	1	0	$P(s)$
b	1	0	0	1	$P(t)$
c	0	1	1	0	$Q(s)$
– – –	– – –	– – –	– – –	– – –	– – –
d	0	1	0	1	$Q(t)$

Analogical proportion as a set of clauses

Since in the proposed approach, an analogical proportion is basically a Boolean formula, it is legitimate to consider what could be inferred from a set of given observations linked through this proportion. Let us start from a simple example to understand our point. Suppose we observe $\neg a, b$ and $\neg c$ and we consider a new d knowing only that d is in analogical proportion with the 3 other items a, b, c. We are faced to the problem of inferring the Boolean value of d. In order to apply the resolution principle, it is necessary to come to clausal forms. It can be checked that the logical expression $a : b :: c : d = ((a \to b) \equiv (c \to d)) \wedge ((b \to a) \equiv (d \to c))$ put in clausal form is equivalent to the conjunction of the following clauses:

$$\mathcal{A} = \{\neg a \vee b \vee c, \neg a \vee b \vee \neg d, a \vee \neg c \vee d, \neg b \vee \neg c \vee d,$$
$$a \vee \neg b \vee \neg c, a \vee \neg b \vee d, \neg a \vee c \vee \neg d, b \vee c \vee \neg d\}$$

Note that we have exactly 8 clauses which cannot be reduced. Each clause is falsified by a pattern of 3 literals for which there does not exist a fourth literal with which to build up a proportion. Thus, the first clause $\neg a \vee b \vee c$ expresses syntactically that $a \;\; \neg b \;\; \neg c$ (i.e., $1 \;\; 0 \;\; 0$ in semantical terms) cannot be analogically completed, while $a \vee \neg b \vee \neg c$ expresses the same w. r. t. $\neg a \;\; b \;\; c$ and $0 \;\; 1 \;\; 1$. Going back to our inference example, it is easy to check the pattern:

$$\frac{\neg a \quad b \quad \neg c \quad a : b :: c : d}{d}$$

by resolution using the 6th clause of the clausal form $\mathcal{A}$ of $a : b :: c : d$. As expected, we have 6 valid inferences which are given in Table 2. They correspond to the 6 lines of the truth table for which the analogical proportion is true. We may notice that inferring the value of d starting from the values of a, b, c and the fact that $a : b :: c : d$ holds can be viewed as an equation solving problem: find d such that $a : b :: c : d$ holds knowing a, b, c.

The Boolean vector case

We want now to extend the previous technique to the case of Boolean vectors $a = (a_1, \ldots, a_n)$ that encode the multiple binary features that describe a situation. Starting from a database (training set) where each piece of data is a row completely informed, we are faced with a learning-like task when we have to consider a new piece of data

Table 2: Valid inferences with an analogical proportion

$\frac{a \;\; b \;\; c \;\; a:b::c:d}{d}$	$\frac{\neg a \;\; \neg b \;\; \neg c \;\; a:b::c:d}{\neg d}$	$\frac{\neg a \;\; \neg b \;\; c \;\; a:b::c:d}{d}$
$\frac{a \;\; \neg b \;\; c \;\; a:b::c:d}{\neg d}$	$\frac{\neg a \;\; b \;\; \neg c \;\; a:b::c:d}{d}$	$\frac{a \;\; b \;\; \neg c \;\; a:b::c:d}{\neg d}$

$d = (d_1, \ldots, d_n)$, partially informed, i.e. where only some features $k(d) = (d_1, \ldots, d_p), p < n$, are known, the values of the missing features $u(d) = (d_{p+1}, \ldots, d_n)$ having to be predicted. In order to perform an inductive step, we adopt the following general pattern:

$$\frac{\forall i \in [1, p], \;\; a_i : b_i :: c_i : d_i}{\forall j \in [p+1, n], \;\; a_j : b_j :: c_j : d_j}$$

It simply means that *if the known part $k(d)$ of d is componentwise in analogical proportion with $k(a), k(b)$ and $k(c)$ then it should be also true for the unknown part $u(d)$ of d.* This is exactly what analogical reasoning is about: we transfer the knowledge we have on the pair (a, b) to the pair (c, d) to predict the missing information about d.
This is obviously a form of reasoning that is not sound, but which may be useful for trying to guess unknown values. As previously seen, we are in a position to safely infer d from $a : b :: c : d$ in the Boolean case, then we are done for the Boolean vector case where we work componentwise.

Example. Let us consider an example where we have 5 binary attributes ($n = 5$) and
$a = (1, 1, 0, 0, 1)$,
$b = (1, 0, 1, 1, 0)$,
$c = (0, 1, 0, 0, 1)$.
We have to predict the missing values for an item
$d = (0, 0, 1, d_4, d_5)$
Here $p = 3$, $k(d) = (0, 0, 1), u(d) = (d_4, d_5)$. With our notation, we see that $k(a) : k(b) :: k(c) : k(d)$ holds for the first three features and we look for the pair of unknown attributes $u(d) = (d_4, d_5)$. Then we consider our previous inference scheme telling us that we should have: $0 : 1 :: 0 : d_4$ and $1 : 0 :: 1 : d_5$. Starting from the clausal definition of $a : b :: c : d$, we use the 6th clause $a_4 \vee \neg b_4 \vee d_4$ which allows to infer d_4, and the 2nd clause $\neg a_5 \vee b_5 \vee \neg d_5$ which allows to infer $\neg d_5$: it means $d_4 = 1$ and $d_5 = 0$ (that we could get from the truth table as well).

The uncertain Boolean vector case

We now assume that the binary value of a feature may be uncertain. Thus the values a_i, b_i, c_i, d_i, are now associated with a certainty level: (a_i, α_i), (b_i, β_i), (c_i, γ_i), (d_i, δ_i). Thus one can compute the certainty λ that an analogical proportion holds for the first p features, as in possibilistic logic (Dubois, Lang, and Prade 1994), namely

$$\lambda = min_{1,p} min(a_i : b_i :: c_i : d_i, \alpha_i, \beta_i, \gamma_i, \delta_i)$$

where $a_i : b_i :: c_i : d_i = 1$ if the analogical proportion holds, and $a_i : b_i :: c_i : d_i = 0$ otherwise.

Then one may consider that we may apply the analogical proportion with certainty λ for completing the missing values for d_j. It amounts to apply possibilistic logic[2] to the possibilistic base

$$K = \{(a_j : b_j :: c_j : d_j, \lambda), (a_j, \alpha_j), (b_j, \beta_j), (c_j, \gamma_j)\}$$

for $p + 1 \leq j \leq n$. Thus, we transfer the uncertainty λ with which the analogical proportion is observed on the first p features, to the extent to which we believe that the analogical proportion still holds on the remaining features.

This treatment should not be confused with the extension of analogical proportion to gradual properties using fuzzy logic (Prade and Richard 2010b).

Related works

It is obvious that analogical inference, considered as a kind of logical reasoning, is unsound (in the classical setting) and then, in some sense, leads to uncertainty. The pioneering works of (Davies and Russell 1987) is an attempt to make analogical inference sound by adding some "side knowledge". This side knowledge (also called "determination rules") are considered as being "minimal" to insure soundness and thus rendering analogical conclusion sound. Unfortunately, these determination rules are rather restrictive and generally not easy to define. A more realistic approach comes from the works of (Qiu 1994) who leaves the classical setting and allows non monotonicity. A non monotonic analogical (first-order) logic system (NAR) is defined, allowing to integrate determination and similarities (since similarities between two items s and t are useful to transfer knowledge from s to t).

Another worth mentioning work is the one by (Kerber and Melis 1996) where two kinds of non monotonic inference processes are defined to model analogical reasoning. Still they provide a formal definition, introducing first the notion of connection between two partial functions: for instance *(population, car)* modeling the fact that the number of inhabitants is linked to the number of registered cars. Using such a "connection", they can infer that similar cities have the same number of registered cars. The uncertainty of the connection is responsible for the uncertainty of the conclusion. Then to overcome the problem of finding a connection or if there is no obvious connection, they introduce the notion of "typical instance" which still allows to infer new information. The non monotonicity still remains due to the fact that the knowledge that we are transfering from the typical instance s to the partially unknown target t is not necessarily relevant to the typical instance. The classical bird Tweety example is a good example of this kind of "uncertain" analogical view of nonmonotonic reasoning.

In the above-mentioned works, the determination rule may be nonmonotonic. In the view explored in this paper, the analogy between two situations (as stated in the first premise of Polya's pattern), is assessed in terms of an analogical proportion on the basis of observed features, and the "jump to a plausible conclusion" amounts to state that the analogy still apply to other partly unknown features. In that respect, it may be of interest to attach to the analogical proportion applied to the new features an uncertainty level that does not just reflect the potential uncertainty of the observations (as discussed in the previous subsection), but rather represent the expected relevance of what has been observed w. r. t. what is predicted.

Concluding remarks

In this note, we have tried first to show how a nonmonotonic consequence relation could be used as a basis for representing analogy between two situations, in agreement with the idea of formal analogical proportions. We have then outlined how such proportions can be used for analogical reasoning purposes and can be combined with uncertainty handling.

Besides, it is worth mentioning that other proportions are available and have been deeply investigated in (Prade and Richard 2010a) (Prade and Richard 2010c) recently. It appears that the general scheme of constraints defining analogical proportion can be slightly modified to give rise to new interesting proportions, capturing other intuitions than the standard analogy. These proportions are viewed as Boolean formula again, with only 6 possibilities for holding as true, as in the case of the analogical proportion. Obviously, what has been done here could be done for these other formulas as well. Interestingly enough, some of these logical proportions have by themselves a nonmonotonic reasoning flavor (Prade and Richard 2010c).

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[2]The resolution rule in possibilistic logic is
$(p \vee q, \alpha), (\neg q \vee r, \beta) \vdash (p \vee r, min(\alpha, \beta))$
See (Dubois, Lang, and Prade 1994).

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