The Role of Syntax in Inductive Inference: A Property-Based Study

Jesse Heyninck[1,3], Richard Booth[2] and Thomas Meyer[3]

[1]Open Universiteit, the Netherlands
[2]Cardiff University, United Kingdom
[3]University of Cape Town and CAIR, South Africa

Abstract
The study of inference operators frequently involves the introduction of properties to which such operators should conform. Amongst other advantages, the property-based approach helps to restrict the range of operators, and to classify and categorise the type of inference being studied. This paper continues this tradition by proposing a number of properties for the class of inductive inference operators. We study the interaction of these properties, both with one another, and with other well-known properties for inductive inference. We also test a number of well-known inductive inference operators against the newly proposed, and some existing properties.

Keywords
inductive inference, lexicographic inference, properties

1. Introduction

In Knowledge Representation, properties (or postulates) provide a standard, and useful, way of studying inference operators. The use of properties allows us to eliminate approaches regarded as undesirable, thereby restricting the attention to the most relevant operators. Viewed as a form of a top-down approach to characterising inference, they are often used in tandem with more bottom-up methods that focus on constructing inference operators. The interaction between these approaches frequently lead to better insights regarding the type of inference being studied. In addition, properties can be used to classify and categorise different approaches to inference, leading to a better understanding of the overall picture.

Two of the best known instances of the use of properties are the AGM properties for belief change [1, 2] and the KLM properties for nonmonotonic inference [3, 4], the latter itself based on the work of Adams [5]. See also the pioneering work of Gabbay [6]. In both cases, the use of properties has had a big impact on the two respective fields, leading to further improvements and clarifications. In fact, it also contributed to the realisation that KLM-style inference and AGM belief change are closely related [7, 8].

In this paper we continue the tradition of employing properties for the study of inference operators understanding, and apply it to a specific kind of nonmonotonic inference operator described by Kern-Isberner et al. [9] as *inductive inference operators.* In particular, we focus on properties that formalize ways in which the syntax of a conditional knowledge base can or should influence the inferences induced by them. More specifically, we make the following contributions:

- We introduce different versions of *equivalence* for inductive inference, point out the relationships between them, and test some well-known forms of inductive inference against them.
- We introduce properties constraining model-based inductive operators to be tightly coupled to the conditional statements provided in a belief base, and show that this is incompatible with one of the notions of equivalence and the property of Syntax Splitting [9].
- Based on the failure of the well-studied form of inductive inference known as *lexicographic inference* [10] to satisfy one of our notions of equivalence, we propose a variant of lexicographic inference that satisfies it.
- We introduce a property of *language independence* for inductive inference and show that a property referred to as *conditional-functional* ensures language independence.

The paper is organized as follows. Section 2 provides the various preliminaries required to present our contributions. Section 3 presents versions of equivalence for inductive inference, and tests this against the newly-added property of being conditional-based. Section 4 presents a version of lexicographic inference that satisfies the notion of pairwise equivalence introduced in the previous section. Section 5 introduces and studies language independence. Section 6 considers related work. Finally, Section 7 concludes and considers future work.

2. Preliminaries

In the following we recall preliminaries on propositional logic, and technical details on inductive inference.

2.1. Propositional Logic

For a set Σ of atoms, let $\mathcal{L}(\Sigma)$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*negation*), $\rightarrow$ (*material implication*) and $\leftrightarrow$ (*material equivalence*). A (classical) *interpretation* (also called *possible world*) ω for a propositional language $\mathcal{L}(\Sigma)$ is a function $\omega : \Sigma \rightarrow \{1, 0\}$ where 1 is understood to denote truth, and 0 to denote falsity. Let $\Omega(\Sigma)$ denote the set of all interpretations for Σ. We simply write Ω if the set of atoms is implicitly given, and similarly for $\mathcal{L}$. An

22nd International Workshop on Nonmonotonic Reasoning, November 2-4, 2024, Hanoi, Vietnam
jesse.heyninck@ou.nl (J. Heyninck); BoothR2@cardiff.ac.uk (R. Booth); tommie.meyer@uct.ac.za (T. Meyer)
https://sites.google.com/view/jesseheyninck (J. Heyninck); https://profiles.cardiff.ac.uk/staff/boothr2 (R. Booth); https://tommiemeyer.org.za (T. Meyer)
0000-0002-3825-4052 (J. Heyninck); 0000-0002-6647-6381 (R. Booth); 0000-0003-2204-6969 (T. Meyer)

interpretation ω *satisfies* (or is a *model* of) an atom $a \in \Sigma$, denoted by $\omega \models a$, if and only if $\omega(a) = 1$. The satisfaction relation $\models$ is extended to formulas in the usual way. As an abbreviation we sometimes identify an interpretation ω with its *complete conjunction*, i. e., if $a_1, \ldots, a_n \in \Sigma$ are those atoms that are assigned $\top$ by ω and $a_{n+1}, \ldots, a_m \in \Sigma$ are those propositions that are assigned $\bot$ by ω we identify ω by $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this). For $X \subseteq \mathcal{L}(\Sigma)$ we also define $\omega \models X$ if and only if $\omega \models A$ for every $A \in X$. Define the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\Sigma) \mid \omega \models X\}$ for every formula or set of formulas X. A formula or set of formulas X_1 *entails* another formula or set of formulas X_2, denoted by $X_1 \models X_2$, if $\mathsf{Mod}(X_1) \subseteq \mathsf{Mod}(X_2)$. Where $\theta \subseteq \Sigma$, and $\omega \in \Omega(\Sigma)$, we denote by ω^θ the restriction of ω to θ, i.e. ω^θ is the interpretation over Σ^θ that agrees with ω on all atoms in θ. Where $\Sigma_i, \Sigma_j \subseteq \Sigma$, $\Omega(\Sigma_i)$ will also be denoted by Ω_i for any $i \in \mathbb{N}$, and likewise $\Omega_{i,j}$ will denote $\Omega(\Sigma_i \cup \Sigma_j)$ (for $i, j \in \mathbb{N}$). Likewise, for some $X \subseteq \mathcal{L}(\Sigma_i)$, we define $\mathsf{Mod}_i(X) = \{\omega \in \Omega_i \mid \omega \models X\}$.

2.2. Reasoning with Nonmonotonic Conditionals

Given a language $\mathcal{L}$, conditionals are objects of the form $(B|A)$ where $A, B \in \mathcal{L}$. The set of all conditionals based on a language $\mathcal{L}$ is defined as: $(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$. We follow the approach of de Finetti [11] who considers conditionals as *generalized indicator functions* for possible worlds or propositional interpretations ω:

$$(B|A)(\omega) = \begin{cases} 1 & : \quad \omega \models A \wedge B \\ 0 & : \quad \omega \models A \wedge \neg B \\ u & : \quad \omega \models \neg A \end{cases} \tag{1}$$

where u stands for *unknown* or *indeterminate*. In other words, a possible world ω *verifies* a conditional $(B|A)$ iff it satisfies both antecedent and conclusion ($(B|A)(\omega) = 1$); it *falsifies, or violates* it iff it satisfies the antecedent but not the conclusion ($(B|A)(\omega) = 0$); otherwise the conditional is *not applicable*, i. e., the interpretation does not satisfy the antecedent ($(B|A)(\omega) = u$). We say that ω *satisfies* a conditional $(B|A)$ iff it does not falsify it, i.e., iff ω satisfies its *material counterpart* $A \to B$. We will look at the semantics of conditionals given both by total preorders (TPOs) $\preceq \subseteq \Omega(\Sigma) \times \Omega(\Sigma)$ and strict partial orders (SPOs) $\prec \subseteq \Omega(\Sigma) \times \Omega(\Sigma)$.[1] As is usual, given a preorder $\preceq$, we denote $\omega \preceq \omega'$ and $\omega' \preceq \omega$ by $\omega \approx \omega'$ and $\omega \preceq \omega'$ and $\omega' \not\preceq \omega$ by $\omega \prec \omega'$. Thus, without loss of generality, the following definition applies to both TPOs and SPOs: given a strict order $\prec$ on possible worlds, representing relative plausibility, we define $A \prec B$ iff for every $\omega' \in \min_\prec(\mathsf{Mod}(B))$ there is an $\omega \in \min_\preceq(\mathsf{Mod}(A))$ such that $\omega \prec \omega'$. This allows for expressing the validity of conditional inferences via stating that $A \mathrel{|\!\!\sim}_\prec B$ iff $(A \wedge B) \prec (A \wedge \neg B)$ [12] for a TPO or SPO. We say that a set $\Delta \subseteq (\mathcal{L}(\Sigma)|\mathcal{L}(\Sigma))$ of conditionals is *consistent* if there is an SPO $\prec$ over $\Omega(\Sigma)$ s.t. $A \mathrel{|\!\!\sim}_\prec B$ for every $(B|A) \in \Delta$. In what follows, we will, for simplicity, always assume a set of conditionals is finite and consistent, and call such a set a *conditional belief base.*

We can *marginalize* total preorders and even inference operators, i.e., restricting them to sublanguages, in a natural way: If $\Theta \subseteq \Sigma$ then any TPO $\preceq$ on $\Omega(\Sigma)$ uniquely induces a *marginalized TPO* $\preceq_{|\Theta}$ on $\Omega(\Theta)$ by setting

$$\omega_1^\Theta \preceq_{|\Theta} \omega_2^\Theta \text{ iff } \omega_1^\Theta \preceq \omega_2^\Theta. \tag{2}$$

Note that on the right hand side of the *iff* condition above $\omega_1^\Theta, \omega_2^\Theta$ are considered as propositions in the superlanguage $\mathcal{L}(\Omega)$. Hence $\omega_1^\Theta \preceq \omega_2^\Theta$ is well defined [13]. SPOs can be marginalized in a similar manner. Similarly, any inference relation $\mathrel{|\!\!\sim}$ on $\mathcal{L}(\Sigma)$ induces a *marginalized inference relation* $\mathrel{|\!\!\sim}_{|\Theta}$ on $\mathcal{L}(\Theta)$ by setting

$$A \mathrel{|\!\!\sim}_{|\Theta} B \text{ iff } A \mathrel{|\!\!\sim} B \tag{3}$$

for any $A, B \in \mathcal{L}(\Theta)$.

An obvious generalisation of total preorders are *ordinal conditional functions (OCFs)*, (also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$. [14]. They express degrees of (im)plausibility of possible worlds and propositional formulas A by setting $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$. A conditional $(B|A)$ is accepted by κ iff $A \mathrel{|\!\!\sim}_\kappa B$ iff $\kappa(A \wedge B) < \kappa(A \wedge \neg B)$. Notice that any OCF induces a TPO on Ω, defined by $\omega_1 \preceq \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$.

2.3. Inductive Inference Operators

In this paper we will be interested in inference operators $\mathrel{|\!\!\sim}_\Delta$ parametrized by a finite conditional belief base Δ. In more detail, such inference operators are *induced by* Δ, in the sense that Δ serves as a starting point for the inferences in $\mathrel{|\!\!\sim}_\Delta$. We call such operators *inductive inference operators*:

Definition 1 ([9]). *An* inductive inference operator (from conditional belief bases) *is a mapping* $\mathbf{C}$ *that assigns to each conditional belief base* $\Delta \subseteq (\mathcal{L}|\mathcal{L})$ *an inference relation* $\mathrel{|\!\!\sim}_\Delta$ *on* $\mathcal{L}$ *that satisfies the following basic requirement of* direct inference*:*

(DI) *If* Δ *is a conditional belief base and* $\mathrel{|\!\!\sim}_\Delta$ *is an inference relation that is induced by* Δ*, then* $(B|A) \in \Delta$ *implies* $A \mathrel{|\!\!\sim}_\Delta B$.

As already indicated in the previous subsection, inference operators can be obtained on the basis of SPOs, TPOs, and OCFs, respectively:

Definition 2. *A* model-based-based inductive inference operator for strict partial orders *is a mapping* $\mathbf{C}^{spo}$ *that assigns to each conditional belief base* Δ *a strict partial order* $\prec_\Delta$ *on* Ω *s.t.* $A \mathrel{|\!\!\sim}_{\prec_\Delta} B$ *for every* $(B|A) \in \Delta$ *(i.e., s.t.* **(DI)** *is ensured). A* model-based inductive inference operator for total preorders $\mathbf{C}^{tpo}$ *is defined similarly, by using a TPO* $\preceq_\Delta$ *instead of an SPO* $\prec_\Delta$.

A model-based inductive inference operator for OCFs *(on* Ω*) is a mapping* $\mathbf{C}^{ocf}$ *that assigns to each conditional belief base* Δ *an OCF* κ_Δ *on* Ω *s.t.* Δ *is accepted by* κ_Δ *(i.e., s.t.* **(DI)** *is ensured).*

Examples of inductive inference operators for OCFs include System Z (also called rational closure, [15, 16], see Sec. 2.4) and c-representations ([17]), whereas lexicographic inference ([10], see Sec. 2.5) is an example of an inductive inference operator for TPOs and System W ([18], see Sec. 2.6) is an example of an inductive inference operator for SPOs.

We now recall a property that has been recently introduced and studied, *syntax splitting* [9]. To define the property of syntax splitting we assume a conditional belief base

[1] A strict partial order is a binary relation that is irreflexive and transitive. A total preorder is a binary relation that is transitive and complete (and therefore reflexive), i.e., $\omega_1 \preceq \omega_2$ or $\omega_2 \preceq \omega_1$ for all ω_1, ω_2.

Δ that can be split into subbases Δ^1, Δ^2 s.t. $\Delta^i \subset (\mathcal{L}_i|\mathcal{L}_i)$ with $\mathcal{L}_i = \mathcal{L}(\Sigma_i)$ for $i = 1,2$ s.t. $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\Sigma_1 \cup \Sigma_2 = \Sigma$, writing $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ whenever this is the case.

Definition 3 (Independence (**Ind**), [9]). *An inductive inference operator* $\mathbf{C}$ *satisfies (***Ind***) if for any* $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ *and for any* $A, B \in \mathcal{L}_i, C \in \mathcal{L}_j$ $(i, j \in \{1,2\}, j \neq i)$,

$$A \mathrel{|\!\sim}_{\Delta} B \text{ iff } A \wedge C \mathrel{|\!\sim}_{\Delta} B$$

Definition 4 (Relevance (**Rel**), [9]). *An inductive inference operator* $\mathbf{C}$ *satisfies (***Rel***) if for any* $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ *and for any* $A, B \in \mathcal{L}_i$ $(i \in \{1,2\})$,

$$A \mathrel{|\!\sim}_{\Delta} B \text{ iff } A \mathrel{|\!\sim}_{\Delta^i} B.$$

Definition 5 (Syntax splitting (**SynSplit**), [9]). *An inductive inference operator* $\mathbf{C}$ *satisfies (***SynSplit***) if it satisfies (***Ind***) and (***Rel***).*

Thus, **(Ind)** requires that inferences from one sublanguage are independent from formulas over the other sublanguage, if the belief base splits over the respective sublanguages. In other words, information on the basis of one sublanguage does not influences inferences made in the other sublanguage. **(Rel)**, on the other hand, restricts the scope of inferences, by requiring that inferences in a sublanguage can be made on the basis of the conditionals in a conditional belief base formulated on the basis of that sublanguage. **(SynSplit)** combines these two properties. It has been shown that System Z satisfies **(Rel)** but not **(Ind)** [9], while lexicographic inference [19] and system W [20] satisfy full **(SynSplit)**.

2.4. System Z

We present system Z as defined by Goldszmidt and Pearl [15] as follows. A conditional $(B|A)$ is tolerated by a finite set of conditionals Δ if there is a possible world ω with $(B|A)(\omega) = 1$ and $(B'|A')(\omega) \neq 0$ for all $(B'|A') \in \Delta$, i.e. ω verifies $(B|A)$ and does not falsify any (other) conditional in Δ. The Z-partitioning (or ordered partition) $OP(\Delta) = (\Delta_0, \ldots, \Delta_n)$ of Δ is defined as:

- $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$;
- $OP(\Delta \setminus \Delta_0) = \Delta_1, \ldots, \Delta_n$.

For $\delta \in \Delta$ we define: $Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$ and $OP(\Delta) = (\Delta_0, \ldots, \Delta_n)$. Finally, the ranking function κ^Z_Δ is defined via: $\kappa^Z_\Delta(\omega) = \max\{Z_\Delta(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $\max \emptyset = -1$. The resulting inductive inference operator $C^{ocf}_{\kappa^Z_\Delta}$ is denoted by C^Z. System Z has been shown to be equivalent to *rational closure* [4], and the two terms are sometimes used interchangeably in the literature.

We now illustrate OCFs in general and system$Z$ in particular with the well-known "Tweety the penguin"-example.

Example 1. *Consider the conditional belief base* $\Delta = \{(f|b), (b|p), (\neg f|p)\}$*, where* b *is intended to represent being a bird,* f *represents being able to fly, and* p *represents being a penguin.* Δ *has the following Z-partitioning:* $\Delta_0 = \{(f|b)\}$ *and* $\Delta_1 = \{(b|p), (\neg f|p)\}$*. This gives rise to the following* κ^Z_Δ*-ordering over the worlds based on the signature* $\{b, f, p\}$*:*

ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ
pbf	2	$pb\overline{f}$	1	$p\overline{b}f$	2	$p\overline{b}\,\overline{f}$	2
$\overline{p}bf$	0	$\overline{p}b\overline{f}$	1	$\overline{p}\overline{b}f$	0	$\overline{p}\overline{b}\,\overline{f}$	0

As an example of a (non-)inference, observe that e.g. $\top \mathrel{|\!\sim}^Z_\Delta \neg p$ *and* $p \wedge f \mathrel{|\!\not\sim}^Z_\Delta b$.

2.5. Lexicographic Inference

We recall lexicographic inference as introduced by Lehmann [10]. For some conditional belief base Δ, the order $\preceq^{\mathsf{lex}}_\Delta$ is defined as follows: Given $\omega \in \Omega$ and $\Delta' \subseteq \Delta$, $V(\omega, \Delta') = |\{(B|A) \in \Delta' \mid (B|A)(\omega) = 0\}|$. Given a set of conditionals Δ partitionend in $OP(\Delta) = (\Delta_0, \ldots, \Delta_n)$, the *lexicographic vector* for a world $\omega \in \Omega$ is the vector $\mathsf{lex}(\omega) = (V(\omega, \Delta_0), \ldots, V(\omega, \Delta_n))$. Given two vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, $(x_1, \ldots, x_n) \preceq^{\mathsf{lex}} (y_1, \ldots, y_n)$ iff there is some $j \leqslant n$ s.t. $x_k = y_k$ for every $k > j$ and $x_j \leqslant y_j$. $\omega \preceq^{\mathsf{lex}}_\Delta \omega'$ iff $\mathsf{lex}(\omega) \preceq^{\mathsf{lex}} \mathsf{lex}(\omega')$. The resulting inductive inference operator $C^{tpo}_{\preceq^{\mathsf{lex}}}$ will be denoted by C^{lex} to avoid clutter.

Example 2 (Example 1 ctd.). *For the Tweety belief base* Δ *as in Example 1 we obtain the following* $\mathsf{lex}(\omega)$*-vectors:*

ω	$\mathsf{lex}(\omega)$	ω	$\mathsf{lex}(\omega)$	ω	$\mathsf{lex}(\omega)$	ω	$\mathsf{lex}(\omega)$
pbf	(0,1)	$pb\overline{f}$	(1,0)	$p\overline{b}f$	(0,2)	$p\overline{b}\,\overline{f}$	(0,1)
$\overline{p}bf$	(0,0)	$\overline{p}b\overline{f}$	(1,0)	$\overline{p}\overline{b}f$	(0,0)	$\overline{p}\overline{b}\,\overline{f}$	(0,0)

The lex*-vectors are ordered as follows:*

$$(0,0) \prec^{\mathsf{lex}} (1,0) \prec^{\mathsf{lex}} (0,1) \prec^{\mathsf{lex}} (0,2).$$

Observe that e.g. $\top \mathrel{|\!\sim}^{\mathsf{lex}}_\Delta \neg p$ *and* $p \wedge f \mathrel{|\!\sim}^{\mathsf{lex}}_\Delta b$.

2.6. System W

System W is a recently introduced inductive inference operator [21, 18] that takes into account the structural information about which conditionals are falsified.

Definition 6 (ξ^j, ξ, preferred structure $\prec^{\mathsf{w}}_\Delta$ on worlds [18]). *For a belief base* $\Delta = \{(B_i|A_i) \mid i \in \{1, \ldots, n\}\}$ *with* $OP(\Delta) = (\Delta^0, \ldots, \Delta^k)$ *and for* $j = 0, \ldots, k$*, the functions* ξ^j *and* ξ *are given by*

$$\xi^j_\Delta(\omega) := \{(B_i|A_i) \in \Delta^j \mid \omega \models A_i \wedge \neg B_i\},$$
$$\xi_\Delta(\omega) := \{(B_i|A_i) \in \Delta \mid \omega \models A_i \wedge \neg B_i\}.$$

If Δ *is clear from the contex, we drop the subindex and write* ξ^j *instead of* ξ^j_Δ *The* preferred structure on worlds *is given by the binary relation* $\prec^{\mathsf{w}}_\Delta \subseteq \Omega \times \Omega$ *defined by, for any* $\omega, \omega' \in \Omega$,

$\omega \prec^{\mathsf{w}}_\Delta \omega'$ *iff there exists an* $m \in \{0, \ldots, k\}$ *such that*

$$\xi^i(\omega) = \xi^i(\omega') \quad \forall i \in \{m+1, \ldots, k\} \text{ and}$$
$$\xi^m(\omega) \subset \xi^m(\omega').$$

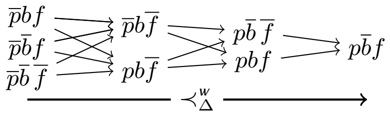
I.e., $\omega \prec^{\mathsf{w}}_\Delta \omega'$ if and only if ω falsifies strictly fewer (in the set-theoretic sense) conditionals than ω' in the Δ^m with the biggest index m where the conditionals falsified by ω and ω' differ. Note that $\prec^{\mathsf{w}}_\Delta$ is a strict partial order [18, Lemma 3].

Definition 7 (System W, $\mathrel{|\!\sim}^{\mathsf{w}}_\Delta$[18]). *Let* Δ *be a belief base and* A, B *be formulas. Then* B *is a* system W inference from A (in the context of Δ), *denoted* $A \mathrel{|\!\sim}^{\mathsf{w}}_\Delta B$*, if for every* $\omega' \in \mathsf{Mod}(A\overline{B})$ *there is an* $\omega \in \mathsf{Mod}(AB)$ *such that* $\omega \prec^{\mathsf{w}}_\Delta \omega'$.

Thus, employing Definition 2, since $\prec^{\mathsf{w}}_{\Delta}$ is a strict partial order, System W is an SPO-based inductive inference operator $C^{\mathsf{w}} : \Delta \mapsto \prec^{\mathsf{w}}_{\Delta}$. In fact, System W strictly lies between System Z and lexicographic inference:

Proposition 1 ([21, 22]). *If A is consistent, then $A \mathrel{|\!\sim}^{z}_{\Delta} B$ implies $A \mathrel{|\!\sim}^{w}_{\Delta} B$ and $A \mathrel{|\!\sim}^{w}_{\Delta} B$ implies $A \mathrel{|\!\sim}^{lex}_{\Delta} B$, but not vice versa.*

Example 3 (Example 1 ctd.). *The belief base Δ from Ex. 1 induces the $\prec^{\mathsf{w}}_{\Delta}$ below. We can entail $pb \mathrel{|\!\sim}^{w}_{\Delta} \overline{f}$ as the verifying world $pb\overline{f}$ is $\prec^{\mathsf{w}}_{\Delta}$-preferred to the only falsifying world pbf, i.e., $pb\overline{f} \prec^{\mathsf{w}}_{\Delta} pbf$.*



3. Equivalence in Conditional Reasoning

We first define some preliminaries regarding equivalence. Two conditionals $(B_1|A_1)$ and $(B_2|A_2)$ are *equivalent* iff every world has the same attitude to both conditionals: i.e. $(B_1|A_1)(\omega) = (B_2|A_2)(\omega)$ for every $\omega \in \Omega$. This is equivalent to $A_1 \equiv A_2$ and $B_1 \wedge A_1 \equiv B_2 \wedge A_2$. Notice that this implies that for any TPO or SPO $\preceq$, $A_1 \mathrel{|\!\sim}_{\preceq} B_1$ iff $A_2 \mathrel{|\!\sim}_{\preceq} B_2$. We write $(B_1|A_1) \equiv (B_2|A_2)$ in that case.

We now define the following kinds of equivalence for conditional knowledge bases:

Definition 8. *Two conditional knowledge bases Δ_1 and Δ_2 are:*

- bijective pairwise equivalent *if there is a bijection $f : \Delta_1 \to \Delta_2$ s.t. $\delta \equiv f(\delta)$ for every $\delta \in \Delta_1$;*
- pairwise equivalent *if for every $\delta_1 \in \Delta_1$ there is some $\delta_2 \in \Delta_2$ s.t. $\delta_1 \equiv \delta_2$, and vice versa;*
- globally equivalent *if for every tpo $\preceq$, Δ_1 is valid w.r.t. $\preceq$ iff Δ_2 is valid w.r.t. $\preceq$.*

The intuition behind these notions is the following: bijective pairwise equivalence requires that two sets of conditionals have the same size, and that every conditional in one set is equivalent to a conditional in the other set. Pairwise equivalence requires that for every conditional in the first set Δ_1, we can find an equivalent conditional in Δ_2, and vice versa, but does not require these sets to have the same size. Finally, global equivalence merely requires that Δ_1 and Δ_2 have the same *semantic* structure, in the sense that they are valid w.r.t. the same TPOs.

These notions are strictly hierarchical:

Proposition 2. *If Δ_1 and Δ_2 are bijectively pairwise equivalent, they are pairwise equivalent, and if they are pairwise equivalent, they are globally equivalent.*

Proof. The implication from bijectively pairwise equivalent to pairwise equivalent is immediate. Suppose Δ_1 and Δ_2 are pairwise equivalent and that Δ_1 is valid w.r.t. $\preceq$. Consider some $(B_2|A_2) \in \Delta_2$. Then with pairwise equivalence, there is some $(B_1|A_1) \in \Delta_1$ s.t. $(B_1|A_1) \equiv (B_2|A_2)$. Since Δ_1 is valid w.r.t. $\preceq$, $A_1 \mathrel{|\!\sim}_{\preceq} B_1$ and thus also $A_2 \mathrel{|\!\sim}_{\preceq} B_2$. □

The following example shows global equivalence does not imply pairwise equivalence:

Example 4. *Consider $\Delta_1 = \{(q|p), (r|p)\}$ and $\Delta_2 = \{(q \wedge r|p)\}$. Then clearly Δ_1 and Δ_2 are globally equivalent but not pairwise equivalent.*

The following example shows pairwise equivalence does not imply bijective pairwise equivalence:

Example 5. *Consider $\Delta_1 = \{(q|p)\}$ and $\Delta_2 = \{(q|p), (q \wedge p|p)\}$. Then clearly Δ_1 and Δ_2 are pairwise equivalent but not bijectively so.*

The following properties express that an inductive inference operator satisfies a given notion of equivalence:

Definition 9. *Let an inductive inference operator $\mathbf{C}$ be given. Then $\mathbf{C}$:*

- satisfies bijective pairwise equivalence *if for any two bijective pairwise equivalent knowledge bases Δ_1 and Δ_2, $\mathbf{C}(\Delta_1) = \mathbf{C}(\Delta_2)$.*
- satisfies pairwise equivalence *if for any two pairwise equivalent knowledge bases Δ_1 and Δ_2, $\mathbf{C}(\Delta_1) = \mathbf{C}(\Delta_2)$.*
- satisfies global equivalence *if for any two pairwise globally equivalent knowledge bases Δ_1 and Δ_2, $\mathbf{C}(\Delta_1) = \mathbf{C}(\Delta_2)$.*

In other words, an inductive inference operator $\mathbf{C}$ satisfies [bijective] pairwise [global] equivalence if for any [bijective] pairwise [globally] equivalent knowledge bases Δ_1 and Δ_2, $A \mathrel{|\!\sim}_{\Delta_1} B$ iff $A \mathrel{|\!\sim}_{\Delta_2} B$. Notice that satisfying global equivalence is the strongest property, and satisfying bijective pairwise equivalence the weakest (this is an immediate consequence of Proposition 2).

We now commence the study of the satisfaction of equivalence for the inductive inference operators introduced above, moving from strongest result to weakest result. The first result concerns System Z:

Proposition 3. *System Z satisfies global equivalence.*

Proof Sketch. Pearl [16] showed that κ^{Z}_{Δ} has the following property: for any $\preceq$ s.t. Δ is valid w.r.t. $\preceq$ and for any $\omega_1, \omega_2 \in \Omega$, $\kappa^{Z}_{\Delta}(\omega_1) < \kappa^{Z}_{\Delta}(\omega_2)$ implies $\omega_1 \prec \omega_2$. Now, if some Δ_1 and Δ_2 are globally equivalent, they have the same TPOs w.r.t. which they are valid, and thus $\kappa^{Z}_{\Delta_1} = \kappa^{Z}_{\Delta_2}$. □

We now move to System W, showing it satisfies pairwise equivalence. We first show some preliminary results The first result shows that *tolerance* is satisfied by pairwise equivalent conditional knowledge bases.

Lemma 1. *Let some conditional δ and two pairwise equivalent conditional knowledge bases Δ_1 and Δ_2 be given. Then Δ_1 tolerates δ iff Δ_2 tolerates δ.*

Proof. Suppose Δ_1 tolerates δ. Then there is an $\omega \in \Omega$ s.t. $\omega(\delta) = 1$ and for every $\delta_1 \in \Delta_1$, $\omega(\delta_1) \neq 0$. Consider some $\delta_2 \in \Delta_2$. Since Δ_1 and Δ_2 are pairwise equivalent, there is a $\delta_1 \in \Delta_1$ s.t. $\delta_1 \equiv \delta_2$. As $\omega(\delta_1) \neq 0$, also $\omega(\delta_2) \neq 0$. Thus, Δ_2 tolerates δ. □

The next result builds on Lemma 1 to show that two pairwise equivalent conditional knowledge bases generate identical ordered partitions (up to pairwise equivalence).

Lemma 2. *Let two pairwise equivalent conditional knowledge bases Δ_1 and Δ_2 with $OP(\Delta_i) = (\Delta_i^1, \ldots, \Delta_i^{n_i})$ (for $i = 1, 2$) be given. Then $n_1 = n_2$ and Δ_1^i is pairwise equivalent with Δ_2^i for $1 \leqslant i \leqslant n_1$.*

Proof Sketch. This is shown by an easy induction on n_i (using Lemma 1). □

Lemma 2 can then be used to prove that System W satisfies pairwise equivalence.

Proposition 4. *System W satisfies pairwise equivalence.*

Proof. Suppose Δ_1 and Δ_2 are pairwise equivalent. Let $OP(\Delta_i) = (\Delta_i^1, \ldots, \Delta_i^{n_i})$ (for $i = 1, 2$). Then with Lemma 2, $n_1 = n_2$. We therefore denote $n_1 (= n_2)$ by n.

We first show the following (†): if Δ_1 and Δ_2 are pairwise equivalent, then for any $\omega \in \Omega$,

$$\xi^j_{\Delta_1}(\omega) = \{\delta_1 \in \Delta_1 \mid \delta_1 \equiv \delta_2 \text{ for some } \delta_2 \in \xi^j_{\Delta_2}(\omega)\}.$$

This is shown as follows. Suppose $\delta_1 \in \xi^j_{\Delta_1}$. Then there is some $\delta_2 \in \Delta_2^j$ (in view of Lemma 2) s.t. $\delta_2 \equiv \delta_1$, which implies $\delta_2 \in \xi^j_{\Delta_2}$. Thus, $\xi^j_{\Delta_1}(\omega) \subseteq \{\delta_1 \in \Delta_1 \mid \delta_1 \equiv \delta_2 \text{ for some } \delta_2 \in \xi^j_{\Delta_2}(\omega)\}$. Furthermore, for every $\delta_2 \in \xi^j_{\Delta_2}$ there is a $\delta_1 \in \xi^j_{\Delta_1}(\omega)$ s.t. $\delta_1 \equiv \delta_2$. Thus, $\xi^j_{\Delta_1}(\omega) \supseteq \{\delta_1 \in \Delta_1 \mid \delta_1 \equiv \delta_2 \text{ for some } \delta_2 \in \xi^j_{\Delta_2}(\omega)\}$

We now show that (‡): for any pairwise equivalent Δ_1 and Δ_2 and any $\omega_1, \omega_2 \in \Omega$, $\xi_{\Delta_1}(\omega_1) \subseteq \xi_{\Delta_1}(\omega_2)$ implies $\xi_{\Delta_2}(\omega_1) \subseteq \xi_{\Delta_2}(\omega_2)$. Indeed, suppose that $\delta_2 \in \xi_{\Delta_2}(\omega_1)$. Then there is some $\delta_1 \in \Delta_2$ s.t. $\delta_1 \equiv \delta_1$. With †, $\delta_1 \in \xi_{\Delta_1}(\omega_1)$ and thus $\delta_1 \in \xi_{\Delta_1}(\omega_2)$. But then with †, $\delta_2 \in \xi_{\Delta_2}(\omega_2)$.

We now show that for any $\omega_1, \omega_2 \in \Omega$, $\omega_1 \prec^w_{\Delta_1} \omega_2$ iff $\omega_1 \prec^w_{\Delta_1} \omega_2$. Suppose for this that $\omega_1 \prec^w_{\Delta_1} \omega_2$, i.e. there is some $1 \leqslant k \leqslant n$ s.t. $\xi^j_{\Delta_1}(\omega_1) = \xi^j_{\Delta_1}(\omega_2)$ for every $j > k$, and $\xi^k_{\Delta_1}(\omega_1) \subset \xi^k_{\Delta_1}(\omega_2)$. With ‡, $\xi^j_{\Delta_2}(\omega_1) = \xi^j_{\Delta_2}(\omega_2)$ for every $j > k$, and $\xi^k_{\Delta_2}(\omega_1) \subset \xi^k_{\Delta_2}(\omega_2)$. Thus, $\omega_1 \prec^w_{\Delta_2} \omega_2$. Altogether, this shows that $A \mathrel{|\!\sim}^{W}_{\Delta_1} B$ iff $A \mathrel{|\!\sim}^{W}_{\Delta_2} B$. □

We'll see in the next section (Proposition 9) that a similar result can not be obtained for global equivalence, though.

We finally turn to lexicographic inference. Perhaps somewhat surprisingly, we observe that lexicographic inference does not even satisfy pairwise equivalence:

Proposition 5. *Lexicographic inference does not satisfy pairwise equivalence.*

Proof. Consider the following conditional knowledge bases:

$$\Delta_1 = \{(p|q), (r|q)\}$$
$$\Delta_2 = \Delta_1 \cup \{(r \land q|q)\}.$$

Δ_1 and Δ_2 are pairwise equivalent. It is easy to see that $OP(\Delta_1) = (\Delta_1)$ and $OP(\Delta_2) = (\Delta_2)$. Thus, the lexicographic vectors for the worlds $\overline{p}qr$ and $pq\overline{r}$ are determined as follows:

$V(\overline{p}qr, \Delta_1) = 1, V(\overline{p}qr, \Delta_2) = 1$, since $\overline{p}qr \models q \land \neg p$;
$V(pq\overline{r}, \Delta_1) = 1, V(pq\overline{r}, \Delta_2) = 2$, since $pq\overline{r} \models q \land \neg r \land \neg(r \land q)$.

This means that $\overline{p}qr \approx^{\mathsf{lex}}_{\Delta_1} pq\overline{r}$ whereas $\overline{p}qr \prec^{\mathsf{lex}}_{\Delta_2} pq\overline{r}$. □

Notice that the example used in this proof is also a violation of the property of Cut when applied to conditionals: if $A \mathrel{|\!\sim}_{\Delta} B$ then $\mathbf{C}(\Delta \cup \{(B|A)\}) \subseteq \mathbf{C}(\Delta)$. This rule says that if we add a conditional to Δ that is already inferred on the basis of Δ - such as $(r \land q|q)$ being added to Δ_1 above - then this should not lead to the inference of any new conditionals. In the above we have, e.g., $q \land (p \leftrightarrow \neg r) \mathrel{|\!\not\sim}_{\Delta_1} r$ but $q \land (p \leftrightarrow \neg r) \mathrel{|\!\sim}_{\Delta_2} r$.

Despite violating pairwise equivalence, lexicographic inference satisfies bijective pairwise equivalence:

Proposition 6. *Lexicographic inference satisfies bijective pairwise equivalence.*

Proof. This is immediate from the fact that for any two bijective pairwise equivalent Δ_1 and Δ_2, $V(\omega, \Delta_1) = V(\omega, \Delta_2)$ (for any $\omega \in \Omega(\Sigma)$). □

Arguably, the failure of lexicographic inference to satisfy pairwise equivalence is undesirable, as it means that the number of (equivalent) conditionals in a knowledge base has an effect on the inferences from the knowledge base. To overcome this defect, we will propose a variant of lexicographic inference that avoids this in Section 4.

3.1. Satisfaction of Global Equivalence and Syntax Splitting

We now show a more general result that shows that the satisfaction of global equivalence is perhaps too strong of a requirement, in the sense that it is incompatible with another property deemed desirable for inductive inference operators, namely syntax splitting. To show this, we will assume another property, namely *conditional-basedness*, which expresses that worlds that have exactly the same attitudes w.r.t. the inducing set of condintionals should not be distinguished. Intuitively, in inductive inference, the only information that is relevant is the set of conditionals the inductive inference operator is based on.

Definition 10. *A model-based inductive inference operator* $\mathbf{C}$ *for TPOs is* conditional-based *if, for any* $\omega_1, \omega_2 \in \Omega$*, if* $(\delta)(\omega_1) = (\delta)(\omega_2)$ *for every* $\delta \in \Delta$ *then* $\omega_1 \approx_\Delta \omega_2$.

A similar property can be defined for model-based inductive inference operators on SPOs and OCFs.

Notice that this is a rather harmless property, in the sense that any of the inductive inference relations studied in this paper satisfy it:

Proposition 7. *System Z, lexicographic inference and System W are conditional-based.*

We can now show that, in the context of conditional-based inductive inference operators, global equivalence and **Ind** are jointly incompatible.

Proposition 8. *There exists no conditional-based inductive inference operation that satisfies global equivalence and satisfies* **(Ind)**.

Proof. Suppose that $\mathbf{C}$ satisfies global equivalence and satisfies syntax splitting.

Consider first $\Delta_1 = \{(a|\top), (b|\top)\}$. With **(DI)**, $\top \mathrel{|\!\sim}^{\mathbf{C}}_{\Delta_1} a$ (which implies $ab \prec \omega$ for any $\omega \in \Omega \setminus \{ab\}$), and likewise, $\top \mathrel{|\!\sim}^{\mathbf{C}}_{\Delta_1} b$. Then since $\Delta_1 = \{(a|\top)\} \bigcup_{\{a\},\{b\}} \{(b|\top)\}$, $\top \land \neg b \mathrel{|\!\sim}^{\mathbf{C}}_{\Delta_1} a$ by **(Ind)**. This means that $a\overline{b} \prec^{\mathbf{C}}_{\Delta_1} \overline{a}\overline{b}$. With symmetry, we establish that $\overline{a}b \prec^{\mathbf{C}}_{\Delta_1} \overline{a}\overline{b}$.

Consider now $\Delta_2 = \{(a \wedge b|\top)\}$. Notice that Δ_1 and Δ_2 are globally equivalent. Thus, since $\mathbf{C}$ satisfies global equivalence, $\prec^{\mathbf{C}}_{\Delta_1} = \prec^{\mathbf{C}}_{\Delta_2}$. However, as $((a \wedge b|\top))(a\overline{b}) = ((a \wedge b|\top))(\overline{a}b) = ((a \wedge b|\top))(\overline{a}\overline{b}) = 0$, we see that $a\overline{b} \approx^{\mathbf{C}}_{\Delta_2} \overline{a}b \approx^{\mathbf{C}}_{\Delta_2} \overline{a}\overline{b}$, contradiction. ☐

Notice that global equivalence is only incompatible with part of **(SynSplit)**, in particular, with the property of **(Ind)**. Indeed, as system Z satisfies **(Rel)** [13], we see it is possible to satisfy global equivalence and **(Rel)**.

We conclude this section with a result following from Proposition 8 (and the fact that System W satisfies **(SynSplit)** ([20]) and is conditional-based (Proposition 7)).

Proposition 9. *System W does not satisfy global equivalence.*

4. A variant of lexicographic inference that satisfies pairwise equivalence

Obtaining a variant of lexicographic inference that satisfies pairwise equivalence is rather straightforward. Instead of counting which conditionals are violated by a world, we count which conditionals are violated *up to equivalence.* In more detail, we observe that equivalence of conditionals is an equivalence relation over $(\mathcal{L}|\mathcal{L})$, and thus, as usual, we define the equivalence class of a conditional $(B|A)$ as $[(B|A)] = \{(D|C) \in (\mathcal{L}|\mathcal{L}) \mid A \equiv C \text{ and } A \wedge B \equiv C \wedge D\}$. We can now count the violations of conditionals in Δ by ω up to equivalence as:

$$V^{\equiv}(\omega, \Delta) := |\{[(B|A)] \mid (B|A)(\omega) = 0, (B|A) \in \Delta\}$$

It is easy to observe that $V^{\equiv}(\omega, \Delta) \leqslant V(\omega, \Delta)$ for any ω and set of conditionals Δ. We can now define, for Δ with $OP(\Delta) = (\Delta_0, \ldots, \Delta_n)$, $\texttt{lex}^{\equiv}(\omega) = (V^{\equiv}(\omega, \Delta_0), \ldots, V^{\equiv}(\omega, \Delta_n))$. We furthermore let $\omega_1 \preceq^{\mathsf{lex},\equiv}_{\Delta} \omega_2$ iff $\texttt{lex}^{\equiv}(\omega_1) \preceq^{\mathsf{lex}} \texttt{lex}^{\equiv}(\omega_2)$. We denote the corresponding inductive inference relation by $C^{\mathsf{lex},\equiv}$ We illustrate this with an adapted Tweety-Example:

Example 6. *Let* $\Delta = \{(f|b), (b|p), (\neg f|p), (\neg f \wedge p|p)\}$. *Notice that* $(\neg f \wedge p|p) \equiv (\neg f|p)$ *We have the following* `lex`*- and* $\texttt{lex}^{\equiv}$*-vectors:*

ω	lex(ω)	lex$^{\equiv}$(ω)	ω	lex(ω)	lex$^{\equiv}$(ω)
pbf	(0,2)	(0,1)	$pb\overline{f}$	(1,0)	(1,0)
$p\overline{b}f$	(0,3)	(0,2)	$p\overline{b}\,\overline{f}$	(0,1)	(0,1)
$\overline{p}bf$	(0,0)	(0,0)	$\overline{p}b\overline{f}$	(1,0)	(1,0)
$\overline{p}\overline{b}f$	(0,0)	(0,0)	$\overline{p}\overline{b}\,\overline{f}$	(0,0)	(0,0)

We see that e.g. $p\overline{bf} \prec^{\mathsf{lex}}_{\Delta} pbf$ *yet* $p\overline{bf} \approx^{\mathsf{lex},\equiv}_{\Delta} pbf$. *This means that* $p \wedge \neg(b \wedge \neg f) \mid\!\sim^{\mathsf{lex}}_{\Delta} \overline{bf}$ *whereas* $p \wedge \neg(b \wedge \neg f) \mid\!\not\sim^{\mathsf{lex},\equiv}_{\Delta} \overline{bf}$.

We note firstly that this inference relation lies between System Z and lexicographic inference:

Proposition 10. *For any conditional knowledge base* Δ, $A \mid\!\sim^{Z}_{\Delta} B$ *implies* $A \mid\!\sim^{\mathsf{lex},\equiv}_{\Delta} B$ *and* $A \mid\!\sim^{\mathsf{lex},\equiv}_{\Delta} B$ *implies* $A \mid\!\sim^{\mathsf{lex}}_{\Delta} B$

Proof. Immediate from the fact that $\preceq^{\mathsf{lex},\equiv}_{\Delta}$ extends κ^{Z}_{Δ} and $\preceq^{\mathsf{lex}}_{\Delta}$ extends $\preceq^{\mathsf{lex},\equiv}_{\Delta}$ (as $V^{\equiv}(\omega, \Delta) \leqslant V(\omega, \Delta)$ for any ω). ☐

The next proposition show that this inference relation is quite well-behaved in the sense that it satisfies pairwise equivalence and syntax splitting.

Proposition 11. $C^{\mathsf{lex},\equiv}$ *satisfies pairwise equivalence.*

Proof. Immediate from the fact that for any pairwise equivalent Δ_1 and Δ_2, $V^{\equiv}(\omega, \Delta_1) = V^{\equiv}(\omega, \Delta_2)$. ☐

Proposition 12. $C^{\mathsf{lex},\equiv}$ *satisfies SynSplit.*

Proof sketch. The proof is essentially the same as that of Theorem 1 by Heyninck et al. [19], with the exception of Lemma 10, which we adapt to $C^{\mathsf{lex},\equiv}$:

Lemma 3. *Let a conditional belief base* $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ *with its corresponding Z-partition* $(\Delta_0, \ldots, \Delta_n)$ *be given. Then for every* $0 \leqslant i \leqslant n$, $V^{\equiv}(\omega, \Delta_i) = V^{\equiv}(\omega^1, \Delta^1_i) + V^{\equiv}(\omega^2, \Delta^2_i)$.[2]

Proof. Take some $0 \leqslant i \leqslant n$. Since $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$, $(B|A) \in \Delta_i$ iff $(B|A) \in \Delta^1_i$ or $(B|A) \in \Delta^2_i$. Furthermore, since $\Sigma_1 \cap \Sigma_2 = \emptyset$, it cannot be the case that $(B|A) \in \Delta^1_i$ and $(B|A) \in \Delta^2_i$. Observe now that: $V^{\equiv}(\omega, \Delta_i) = |\{[(B|A)] \in \Delta_i \mid \omega \models A \wedge \neg B\}| = |\{[(B|A)] \mid \omega^1 \models A \wedge \neg B \text{ and } (B|A) \in \Delta^1_i\}| \cup |\{[(B|A)] \mid \omega^2 \models A \wedge \neg B \text{ and } (B|A) \in \Delta^2_i\}|$. Thus: $V^{\equiv}(\omega, \Delta_i) = V^{\equiv}(\omega^1, \Delta^1_i) + V^{\equiv}(\omega^2, \Delta^2_i)$, because $\{[(B|A)] \mid \omega^1 \models A \wedge \neg B \text{ and } (B|A) \in \Delta^1_i\} \cap \{[(B|A)] \mid \omega^2 \models A \wedge \neg B \text{ and } (B|A) \in \Delta^2_i\} = \emptyset$. ☐

This completes the proof of Proposition 12. ☐

Furthermore, it should be noticed that, as $\mathbf{C}^{\mathsf{lex},\equiv}$ is a model-based inductive inference operator for strict partial orders, it satisfies all the so-called KLM-postulates, including rational monotony.

5. Language Independence

A final property of inductive inference operators we study is the property of *language independence.* This property intuitively states that inductive inference should be independent of how exactly atoms are expressed. For example, it should not matter whether we represent two atoms as a and b or p and q. More generally, in many cases, atoms can be equivalently represented as complex formulas and vice versa. For example, one can represent "I don't have a dog" by a or $\neg a$. This idea is formalized by Marquis and Schwind [23] by defining a symbol translation as any mapping $\sigma : \Sigma \to \mathcal{L}(\Sigma')$. We can extend such a translation to formulas by simply defining $\sigma(A)$ the formula obtained by substituting any $p \in \Sigma$ by $\sigma(p)$ in A, and for any conditional knowledge base Δ we denote by $\sigma(\Delta)$ the knowledge base obtained by replacing each $(B \mid A)$ in Δ by $(\sigma(B) \mid \sigma(A))$. We restrict attention to a specific class of symbol translations.

[2] Notice that it follows from Fact **??** that, given the Z-partition $(\Delta_1, \ldots, \Delta_n)$ of Δ and $\Sigma_i \subseteq \Sigma$, $\Delta^i_j = \Delta_j \cap (\mathcal{L}_i|\mathcal{L}_i)$ for any $0 \leqslant j \leqslant n$.

Definition 11. *A mapping $\sigma : \Sigma \to \mathcal{L}(\Sigma')$ is a* belief-amount preserving symbol translation *(in short, a BAP-translation) if there is a bijection $\gamma : \Omega(\Sigma) \to \Omega(\Sigma')$ s.t. for every $A \in \mathcal{L}(\Sigma)$ $\mathsf{Mod}(\sigma(A)) = \{\gamma(\omega) \mid \omega \in \mathsf{Mod}(A)\}$.*

The idea is that a symbol translation is a way of translating every atom to a formula such that the images of atoms are semantically equivalent (in the new language) to their originals: every world in the original language corresponds to exactly one world in the translated language.

Example 7. *Consider $\Sigma = \{a, b\}$ and the symbol translation $\sigma(a) = a$ and $\sigma(b) = a \leftrightarrow b$. Then we have the following translations of $\Omega(\Sigma)$:*

$$\begin{array}{rcll} \sigma(ab) & = & a \wedge a \leftrightarrow b & (= ab) \\ \sigma(a\overline{b}) & = & a \wedge \neg(a \leftrightarrow b) & (= a\overline{b}) \\ \sigma(\overline{a}b) & = & \overline{a} \wedge (a \leftrightarrow b) & (= \overline{a}\overline{b}) \\ \sigma(\overline{a}\overline{b}) & = & \overline{a} \wedge \neg(a \leftrightarrow b) & (= \overline{a}b) \end{array}$$

We thus see that the bijection γ with $\gamma(\overline{a}b) = \overline{a}\overline{b}, \gamma(\overline{a}\overline{b}) = \overline{a}b$ and that maps ab and $a\overline{b}$ to their selves is a bijection that ensures σ is a BAP-translation.

We are now ready to state what it means for an inductive inference operator to satisfy language independence—it should be invariant under BAP symbol translation.

Definition 12. *An inductive inference operator $\mathbf{C}$ satisfies* language independence *if for every BAP-translation σ, $A \mathrel{|\!\sim}_{\Delta} B$ iff $\sigma(A) \mathrel{|\!\sim}_{\sigma(\Delta)} \sigma(B)$.*

On the level of TPOs, we obtain the following representation of language independence (variants for OCFs and SPOs are obtained similarly):

Proposition 13. *A model-based inductive inference operator for TPOs $\mathbf{C}$ satisfies language independence if for every BAP-translation $\sigma : \Sigma_1 \to \Sigma_2$, and any conditional belief base Δ over $\mathcal{L}(\Sigma_1)$, $\omega_1 \preceq_{\Delta} \omega_2$ iff $\sigma(\omega_1) \preceq_{\sigma(\Delta)} \sigma(\omega_2)$.*

Proof. Suppose that for every BAP-translation $\sigma : \Sigma_1 \to \mathcal{L}(\Sigma_2)$, and any conditional belief base Δ over Σ_1, $\omega_1 \preceq_{\Delta} \omega_2$ iff $\sigma(\omega_1) \preceq_{\sigma(\Delta)} \sigma(\omega_2)$. Suppose now that $A \mathrel{|\!\sim}_{\Delta} B$, i.e. $\min_{\preceq_{\Delta}}(\mathsf{Mod}(A)) \subseteq \mathsf{Mod}(B)$. As the order of $\preceq_{\Delta}$ is preserved over σ, $\min_{\preceq_{\sigma(\Delta)}}(\mathsf{Mod}(\sigma(A))) = \{\sigma(\omega) \mid \omega \in \min_{\preceq_{\Delta}}(A)\}$. As σ is BAP-translation, $\{\sigma(\omega) \mid \omega \in \min_{\preceq_{\Delta}}(A)\} \subseteq \mathsf{Mod}(\sigma(B))$. Thus, $\min_{\preceq_{\sigma(\Delta)}}(\mathsf{Mod}(\sigma(A))) \subseteq \mathsf{Mod}(\sigma(B))$ which implies $\sigma(A) \mathrel{|\!\sim}_{\sigma(\Delta)} \sigma(B)$. The proof of the opposite direction ($\sigma(A) \mathrel{|\!\sim}_{\sigma(\Delta)} \sigma(B)$ implies $A \mathrel{|\!\sim}_{\Delta} B$) is similar. □

When are inductive inference operators language independent? We delineate a condition that ensures language independence (as well as generalising the property of being conditional-based), which we call conditional-functional. Intuitively, this property requires that an induced consequence relation $\mathbf{C}(\Delta)$ only depends on the attitudes worlds have w.r.t. conditionals. Formally defining this property turned out to be rather intricate, and we do so below in full detail. Intuitively, the idea is that we are interested in inference operators that only depend on vectors $\langle i_1, \ldots, i_n \rangle$ of attitudes of worlds to conditionals.

In more formal detail, we let an *n-dimensional vector mass distribution* (nVMD) for a signature Σ be a function $F : \{1, 0, u\}^n \mapsto \mathbb{N}$ s.t. $\Sigma_{\vec{\alpha} \in \{1,0,u\}^n} F(\vec{\alpha}) = 2^{|\Sigma|}$. Intuitively, a VMD F is a function that keeps track of how many times every vector of attitudes occurs. This can be seen as a placeholder for a conditional knowledge base, in the sense that this is the only information about a conditional knowledge base that should be of interest for a conditional-functional inductive inference operator that looks solely at the attitudes of worlds w.r.t. conditionals. We can now define conditional-funtionality:

Definition 13. *An inductive inference operator $\mathbf{C}$ is* conditional-functional *iff there is a function D that returns, for any n and any nVMD F, a pair $(V_F^D, \sqsubseteq_F^D)$ where:*

1. *$V_F^D \subseteq \{1, 0, u\}^n$,*
2. *$\sqsubseteq_F^D$ is a TPO on V_F^D.*

such that:

- *$\vec{\alpha} \in V_F^D$ implies $F(\vec{\alpha}) > 0$, and*
- *for any permutation σ on $\{1, \ldots, n\}$, $V_{\sigma(F)}^D = \sigma(V_F^D)$ and $\vec{\alpha} \sqsubseteq_{\sigma(F)}^D \vec{\beta}$ iff $\sigma^{-1}(\vec{\alpha}) \sqsubseteq_F^D \sigma^{-1}(\vec{\beta})$*

and such that $\omega_1 \preceq_{\Delta} \omega_2$ iff $\langle \delta_1(\omega_1), \ldots, \delta_n(\omega_1) \rangle \sqsubseteq_{F_{\Delta}}^D \langle \delta_1(\omega_2), \ldots, \delta_n(\omega_2) \rangle$, where $\Delta = \{\delta_1, \ldots, \delta_n\}$ and $F_{\Delta}(\vec{\alpha}) = |\{\omega \mid \vec{\alpha} = \langle \delta_1(\omega), \ldots, \delta_n(\omega) \rangle\}|$.

The intuition behind this definition is the following: $\mathbf{C}$ should depend only on attitudes of worlds w.r.t. conditionals. That is, we should be able to formulate it on basis of the VMD alone. This is formalized by the condition that $\omega_1 \preceq_{\Delta} \omega_2$ iff $\langle \delta_1(\omega_1), \ldots, \delta_n(\omega_1) \rangle \sqsubseteq_{F_{\Delta}}^D \langle \delta_1(\omega_2), \ldots, \delta_n(\omega_2) \rangle$ for some function D generating TPOs over the attitude-vectors that depends only on the VMD. Furthermore, the exact ordering of the conditionals in a conditional knowledge base should not matter. Hence the requirement of invariance under permutations.

Example 8. *We start with the conditional belief base from Example 1 and show how it can be interpreted in terms of a 3VMD. We first recall that the worlds have the following attitudes w.r.t. the conditionals $\delta_1 = (f|b)$, $\delta_2 = (b|p)$ and $\delta_3 = (\neg f|p)$:*

ω	$\omega(\delta_1)$	$\omega(\delta_2)$	$\omega(\delta_3)$	ω	$\omega(\delta_1)$	$\omega(\delta_2)$	$\omega(\delta_3)$
pbf	1	1	0	$pb\overline{f}$	u	0	0
$p\overline{b}f$	1	u	u	$p\overline{b}\,\overline{f}$	u	u	u
$\overline{p}bf$	0	1	1	$\overline{p}b\overline{f}$	u	0	1
$\overline{p}\overline{b}f$	0	u	u	$\overline{p}\overline{b}\,\overline{f}$	u	u	u

This means that we can view this knowledge base as the 3VMD F defined by:

$\vec{\alpha}$	$F(\vec{\alpha})$	$\vec{\alpha}$	$F(\vec{\alpha})$
$\langle 1, 1, 0 \rangle$	1	$\langle u, 0, 0 \rangle$	1
$\langle 1, u, u \rangle$	1	$\langle u, u, u \rangle$	2
$\langle 0, 1, 1 \rangle$	1	$\langle u, 0, 1 \rangle$	1
$\langle 0, u, u \rangle$	1		

and $F(\vec{\alpha}) = 0$ for all remaining $\vec{\alpha} \in \{1, 0, u\}^n$.

Furthermore, system Z for this instance is captured by $D(F) = (V_F^D, \sqsubseteq_F^D)$ with $V_F^D = \{\vec{\alpha} \mid F(\vec{\alpha} > 0\}$ and $\langle u, u, u \rangle, \langle 1, u, u \rangle \sqsubseteq_F^D \langle 0, u, u \rangle, \langle 0, 1, 1 \rangle \sqsubseteq_F^D \langle 1, 1, 0 \rangle, \langle u, 0, 0 \rangle, \langle u, 0, 1 \rangle$.

The next result shows that for TPO-based inductive inference operators, being conditional-functional implies satisfying bijective pairwise-equivalence and language independence.

Proposition 14. *If a TPO-based inductive inference operator* $\mathbf{C}$ *is conditional-functional then it satisfies bijective pairwise equivalence and language independence.*

Proof. We first show *language independence*. For this, consider a set of conditionals $\Delta \subseteq (\mathcal{L}(\Sigma)|\mathcal{L}(\Sigma))$ and a BAP-translation σ with corresponding bijection $\gamma : \Omega(\Sigma) \to \Omega(\Sigma')$. With Proposition 13, we have to show that $\omega_1 \preceq_\Delta \omega_2$ iff $\sigma(\omega_1) \preceq_{\sigma(\Delta)} \sigma(\omega_2)$ for any $\omega_1, \omega_2 \in \Omega(\Sigma)$. First, notice that, as σ is a BAP-translation, $\omega \models A$ iff $\gamma(\omega) \models \sigma(A)$ for any $A \in \mathcal{L}(\Sigma)$. This means that for any $\delta \in \Delta$, $\delta(\omega) = \sigma(\delta)(\gamma(\omega))$. Thus, $\langle \delta_1(\omega), \ldots, \delta_n(\omega)\rangle = \langle \sigma(\delta_1)(\gamma(\omega)), \ldots, \sigma(\delta_n)(\gamma(\omega))\rangle$. As $\mathbf{C}$ is conditional-based, and $\sigma(\omega) = \gamma(\omega)$ for any $\omega \in \Omega(\Sigma)$, this immediately implies that $\omega_1 \preceq_\Delta \omega_2$ iff $\sigma(\omega_1) \preceq_{\sigma(\Delta)} \sigma(\omega_2)$ for any $\omega_1, \omega_2 \in \Omega(\Sigma)$.

Next we show *the satisfaction of bijective pairwise equivalence*. For this, consider two knowledge bases $\Delta_1 = \{\delta_1, \ldots, \delta_n\}$ and $\Delta_2 = \{\delta'_1, \ldots, \delta'_n\}$ that are bijectively pairwise equivalent, where f is the bijection $f : \Delta_1 \mapsto \Delta_2$ s.t. $f(\delta) \equiv \delta$ for every $\delta \in \Delta_1$. Then clearly, for every $\omega \in \Omega(\Sigma)$, $\delta(\omega) = f(\delta)(\omega)$ (by definition of bijective pairwise equivalence). Thus, $\langle \delta_1(\omega), \ldots, \delta_n(\omega)\rangle = \langle f(\delta_1)(\omega), \ldots, f(\delta_n)(\omega)\rangle$. As D is invariant under permutations on $\{1, \ldots, n\}$, $\langle f(\delta_1)(\omega), \ldots, f(\delta_n)(\omega)\rangle$ and $\langle \delta'_1(\omega), \ldots, \delta'_n(\omega)\rangle$ get assigned the same position according to $\sqsubseteq^D$. This concludes the proof. ☐

The final result in this section shows the converse—for TPO-based inductive inference operators, satisfying bijective pairwise equivalence and language independence implies being conditional-functional.

Proposition 15. *If a TPO-based inductive inference operator* $\mathbf{C}$ *satisfies bijective pairwise equivalence and language independence then it is conditional-functional.*

Proof. Given $\mathbf{C}$ satisfying bijective pairwise equivalence and language independence, we must construct D from $\mathbf{C}$ such that $\mathbf{C} = \mathbf{C}_D$. Let F be a VMD of dimension m. We must specify V_F^D and $\sqsubseteq_F^D$. Our strategy will be to construct from F a particular conditional belief base Δ_F of size m and then use $\mathbf{C}(\Delta_F)$ to define V_F^D and $\sqsubseteq_F^D$. As a first step, let $\langle \omega_1, \ldots, \omega_{2^n}\rangle$ be an arbitrary enumeration of the interpretations and let $\langle \vec{\alpha}_1, \ldots, \vec{\alpha}_p\rangle$ be an enumeration of the set of all $\vec{\alpha}$ such that $F(\vec{\alpha}) > 0$, ordered lexicographically under the assumption $0 < u < 1$. We now choose some way to distribute the $\vec{\alpha}_j$ among the interpretations. A concrete way to do this is to set up a function t assigning to each interpretation ω_i a vector $\vec{\alpha}_j$ as follows:

$$t(\omega_i) = \vec{\alpha}_j \text{ where } j \text{ is min. s.t. } \sum_{k \leqslant j} F(\vec{\alpha}_k) \geqslant i$$

In other words, assign $\vec{\alpha}_1$ to the first $F(\vec{\alpha}_1)$ interpretations in $\langle \omega_1, \ldots, \omega_{2^n}\rangle$, then assign $\vec{\alpha}_2$ to the next $F(\vec{\alpha}_2)$ interpretations in the list, and so on. Then let $\Delta_F = \{(B_1|A_1), \ldots, (B_m|A_m)\}$, where, for each $k = 1, \ldots, m$:

$$A_k = \bigvee \{\omega_i \mid k^{\text{th}} \text{ element of } t(\omega_i) \text{ is 0 or 1}\}$$

$$B_k = \bot \vee \bigvee \{\omega_i \mid k^{\text{th}} \text{ element of } t(\omega_i) \text{ is 1}\}$$

Now let $\mathrel{|\!\sim}^* = \mathbf{C}(\Delta_F)$ with $\preceq_*$ its associated TPO. Then V_F^D and $\sqsubseteq_F^D$ are specified as follows:

$$V_F^D = \{\vec{\alpha}_j \mid \omega_i \not\mathrel{|\!\sim}^* \bot \text{ for some } \omega_i \text{ s.t. } t(\omega_i) = \vec{a}_j\}$$

and, for any $j_1, j_2 \in \{1, \ldots, p\}$,

$$\vec{\alpha}_{j_1} \sqsubseteq_F^D \vec{\alpha}_{j_2} \quad \text{iff} \quad \omega_{i_1} \preceq_* \omega_{i_2} \text{ for some } i_1, i_2 \text{ s.t. } t(\omega_{i_1}) = \vec{\alpha}_{j_1} \text{ and } t(\omega_{i_2}) = \vec{\alpha}_{j_2}$$

We now show that the conditions from Definition 13 are satisfied:

1. [$\vec{\alpha} \in V_F^D$ *implies* $F(\vec{\alpha}) > 0$]: clear from construction.

2. [*for any permutation* σ *on* $\{1, \ldots, n\}$, $V_{\sigma(F)}^D = \sigma(V_F^D)$ *and* $\vec{\alpha} \sqsubseteq_{\sigma(F)}^D \vec{\beta}$ *iff* $\sigma(\vec{\alpha}) \sqsubseteq_F^D \sigma(\vec{\beta})$]: Let σ be a permutation on $\{1, \ldots, n\}$, i.e. there is a bijection $f : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$ s.t. for every $\langle \alpha_1, \ldots, \alpha_n\rangle \in \{1, 0, u\}^n$, $\sigma(\langle \alpha_1, \ldots, \alpha_n\rangle) = \langle \alpha_{f(1)}, \ldots, \alpha_{f(n)}\rangle$. Define $\sigma(F)$ by $\sigma(F)(\langle \alpha_1, \ldots, \alpha_n\rangle) = F(\sigma^{-1}(\langle \alpha_1, \ldots, \alpha_n\rangle))$. We now show the construction is invariant under σ. Let t' be the assignment of vectors to worlds relative to the VMD $\sigma(F)$. We start by defining a bijection $\gamma : \Omega(\Sigma) \mapsto \Omega(\Sigma)$ s.t. for every $\omega \in \Omega(\Sigma)$, $\gamma(\omega) = \omega'$ implies that $t(\omega) \in \sigma^{-1}(t'(\omega'))$. Notice that by the definition of t' and the construction above, such a bijection is guaranteed to exist (but might not be unique). Intuitively, γ maps every world ω to one of its σ-counterparts ω' (i.e. ω corresponds to the vector $\vec{\alpha}$ and ω' corresponds to the vector $\sigma(\vec{\alpha})$). Define now $\tau : \Sigma \mapsto \mathcal{L}(\Sigma)$ by $\tau(a) = \bigvee\{\gamma(\omega) \mid \omega \in \mathsf{Mod}(a)\}$. It can be easily checked that this is a BAP-translation. Furthermore, it can be easily seen that: (†): $\Delta_{\sigma(F)}$ is bijectively pairwise equivalent to $\tau(\Delta_F)$

We now show that $V_{\sigma(F)}^D = \sigma(V_F^D)$. Suppose first that $\vec{\alpha} \in V_{\sigma(F)}^D$, i.e. $\omega \not\mathrel{|\!\sim}^{\mathbf{C}}_{\Delta_F} \bot$ for some ω s.t. $t(\omega) = \vec{\alpha}$. As τ is a BAP-translation and $\mathbf{C}$ is language independent, satisfies bijective pairwise equivalence and in view of (†), $\tau(\omega) \not\mathrel{|\!\sim}^{\mathbf{C}}_{\sigma(\Delta_F)} \bot$ and thus $\tau(\omega) \in \sigma(V_F^D)$. As $\tau(\omega) = \gamma(\omega)$ and $t'(\gamma(\omega)) = \sigma(t(\omega)) = \sigma(\vec{\alpha})$ (by construction of γ), we see that $\sigma(\vec{\alpha}) \in \sigma(V_F^D)$. The opposite direction is similar.

We now show that $\vec{\alpha} \sqsubseteq_{\sigma(F)}^D \vec{\beta}$ iff $\sigma(\vec{\alpha}) \sqsubseteq_F^D \sigma(\vec{\beta})$. Suppose first that $\vec{\alpha} \sqsubseteq_{\sigma(F)}^D \vec{\beta}$, i.e. there are some $\omega_{\vec{\alpha}}, \omega_{\vec{\beta}}$ with $t(\omega_x) = x$ for $x = \vec{\alpha}, \vec{\beta}$ and $\omega_{\vec{\alpha}} \prec^{\mathbf{C}}_{\Delta_F} \omega_{\vec{\beta}}$. As τ is a BAP-translation and $\mathbf{C}$ is language independent, satisfies bijective pairwise equivalence and in view of (†), $\gamma(\omega_{\vec{\alpha}}) \prec^{\mathbf{C}}_{\sigma(\Delta_F)} \gamma(\omega_{\vec{\beta}})$. As $t'(\gamma(\omega_x)) = \sigma(t(\omega_x)) = \sigma(\vec{x})$ for $x = \vec{\alpha}, \vec{\beta}$ (by construction of γ), we see that $\sigma(\vec{\alpha}) \sqsubseteq_F^D \sigma(\vec{\beta})$. The opposite direction is similar.

3. [$\omega_1 \preceq_\Delta \omega_2$ *iff* $\langle \delta_1(\omega_1), \ldots, \delta_n(\omega_1)\rangle \sqsubseteq_{F_\Delta}^D \langle \delta_1(\omega_2), \ldots, \delta_n(\omega_2)\rangle$, *where* $\Delta = \{\delta_1, \ldots, \delta_n\}$ *and* $F_\Delta(\vec{\alpha}) = |\{\omega \mid \vec{\alpha} = \delta_1(\omega), \ldots, \delta_n(\omega)\}|$]: This can be easily seen by observing the following: for every $\omega \in \Omega(\Sigma)$, $\langle \delta_1(\omega), \ldots, \delta_m(\omega)\rangle = t(\omega)$. Indeed, consider $\delta_i = (\bot \vee \bigvee \Omega_1 | \bigvee \Omega_2)$ for some $i = 1, \ldots, m$, where Ω_1 and Ω_2 are the sets of worlds as defined for Δ_F above. We consider three cases: (i.) $\delta_m(\omega) = 1$. This can only happen if $\omega \in \Omega_1 \cap \Omega_2$, which implies that the i^{th} element of $t(\omega)$ is 1. (ii.) $\delta_m(\omega) = 0$. This can only happen if $\omega \in \Omega_1 \setminus \Omega_2$, which implies that the i^{th} element of $t(\omega)$ is 0. (iii.) This can only happen if $\omega \notin \Omega_1$, which implies the i^{th} element of $t(\omega)$ is u. As this exhausts all the options, this is sufficient to show the claim.

We must now show that $\mathbf{C}_D = \mathbf{C}$. So let $\{\delta_1, \ldots, \delta_m\}$ be a conditional belief base. For each $i = 1, \ldots, 2^n$, let $\vec{b}_i = \langle \delta_1(\omega_i), \ldots, \delta_m(\omega_i)\rangle$ and let γ be a permutation on $\{1, \ldots, 2^n\}$ such that the sequence $\langle \vec{b}_{\gamma(1)}, \ldots, \vec{b}_{\gamma(2^n)}\rangle$ is sorted lexicographically. Let σ be the BAP-translation corresponding to γ. Then, since $\mathbf{C}$ satisfies language independence, we have: $A \mathrel{|\!\sim}_\Delta B$ iff $\sigma(A) \mathrel{|\!\sim}_{\sigma(\Delta)} \sigma(B)$ ☐

Based on their construction, we strongly expect that system Z, lexicographic inference and system W are conditional functional, but a rigorous proof is left to the next version of this work.

6. Related Work

While there are several works related to the work we have done in this paper [24, 13, 25], the work of Weydert [24] is perhaps the closest. Weydert suggested several properties in his study of default reasoning that share strong commonalities with some of the properties discussed in our work. For example, *global logical invariance* is rather similar to the satisfaction of global equivalence, even though he does not define (as far as we could see) in a formally precise manner when two sets of conditionals are equivalent. Furthermore, *strong irrelevance* is very similar to *relevance* and *representation independence* is very similar to language independence (with the main differences being induced by the differences in the assumptions of the framework of Weydert, such as allowing for languages generated on the basis of infinite boolean algebras, and allowing for rankings over rational numbers). Furthermore, he shows, in his *exceptional inheritance paradox*, that no consistent default inference notion (his version of an inductive operator)can satisfy logicality, exceptional inheritance and global logical invariance. This is a slightly different but still quite similar result to our proposition 8. Essentially, we assume syntax splitting whereas he assumes logicality (which means that the basic KLM-properties are satisfied) and exceptional inheritance. Exceptional inheritance states that $\{\phi, \neg\psi\} \mid\!\sim_{\{(\psi|\phi),(\psi'|\phi)\}} \psi'$ if "ψ and ψ' are logically independent given ϕ", although the concept of logical independence is not precisely formalised. We note as a further difference that he does not study System Z, System W and lexicographic inference as we.

7. Conclusion

This paper continues a tradition of studying inductive inference operators using properties. Table 1 summarizes the main findings of our work. More specifically, it considers the inductive inference operators System Z, System W, lexicographic inference, and the variation of lexicographic inference introduced in Section 4, and shows, for each of them, whether or not they satisfy the properties of Independence, Relevance, Global Equivalence, Pairwise Equivalence, Conditional-Based, and Language Independence.

	System Z	System W	Lex	Lex$^{\equiv}$
Independence	× ([9])	∨ ([20])	∨ ([19])	∨
Relevance	∨ ([9])	∨ ([20])	∨ ([19])	∨
Global Eq.	∨	×	×	×
Pairwise Eq.	∨	∨	×	∨
Bij. Pairwise Eq.	∨	∨	∨	∨
Cond.-based	∨	∨	∨	∨

Table 1
Summary of the properties studied in this paper, where previous shown results occur with the respective reference.

We see several avenues for future work. An obvious direction is to study other inductive inference operators, such as c-representations [17], relevant closure [26], disjunctive rational closure [27] or Weydert's many System J-variants [24]. Another avenue for future work is to see whether these postulates can also be helpful in characterising inductive inference operators.

References

[1] C. Alchourrón, P. Gärdenfors, D. Makinson, On the logic of theory change: Partial meet contraction and revision functions, Journal of Symbolic Logic 50 (1985) 510–530.

[2] H. Katsuno, A. Mendelzon, Propositional knowledge base revision and minimal change, Artificial Intelligence 3 (1991) 263–294.

[3] S. Kraus, D. Lehmann, M. Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, Artificial Intelligence 44 (1990) 167–207.

[4] D. Lehmann, M. Magidor, What does a conditional knowledge base entail?, Artificial intelligence 55 (1992) 1–60.

[5] E. Adams, Probability and the logic of conditionals, in: J. Hintikka, P. Suppes (Eds.), Aspects of inductive logic, North-Holland, 1966, pp. 265––316.

[6] D. M. Gabbay, Theoretical foundations for non-monotonic reasoning in expert systems, in: K. R. Apt (Ed.), Logics and Models of Concurrent Systems, Springer Berlin Heidelberg, Berlin, Heidelberg, 1985, pp. 439–457.

[7] P. Gärdenfors, Belief revision and nonmonotonic logic: Two sides of the same coin?, in: Proc. JELIA'90, Springer, 1991, pp. 52–54.

[8] P. Gärdenfors, D. Makinson, Nonmonotonic inference based on expectations, Artificial Intelligence 65 (1994) 197–245.

[9] G. Kern-Isberner, C. Beierle, G. Brewka, Syntax splitting= relevance+ independence: New postulates for nonmonotonic reasoning from conditional belief bases, in: Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning, volume 17, 2020, pp. 560–571.

[10] D. Lehmann, Another perspective on default reasoning, Annals of Mathematics and Artificial Intelligence 15 (1995) 61–82.

[11] B. de Finetti, La prévision, ses lois logiques et ses sources subjectives, 1937. English translation in *Studies in Subjective Probability*, ed. H. Kyburg and H.E. Smokler, 1974, 93–158. New York: Wiley & Sons.

[12] D. Makinson, General theory of cumulative inference, in: International Workshop on Non-Monotonic Reasoning (NMR), Springer, 1988, pp. 1–18.

[13] G. Kern-Isberner, G. Brewka, Strong Syntax Splitting for Iterated Belief Revision, in: C. Sierra (Ed.), Proceedings International Joint Conference on Artificial Intelligence, IJCAI 2017, ijcai.org, 2017, pp. 1131–1137.

[14] W. Spohn, Ordinal conditional functions: A dynamic theory of epistemic states, in: Causation in decision, belief change, and statistics, Springer, 1988, pp. 105–134.

[15] M. Goldszmidt, J. Pearl, Qualitative probabilities for default reasoning, belief revision, and causal modeling, Artificial Intelligence 84 (1996) 57–112.

[16] J. Pearl, System Z: a natural ordering of defaults with tractable applications to nonmonotonic reasoning, in: Proceedings of the 3rd conference on Theoretical aspects of reasoning about knowledge, 1990, pp. 121–135.

[17] G. Kern-Isberner, Handling conditionals adequately in uncertain reasoning and belief revision, Journal of Applied Non-Classical Logics 12 (2002) 215–237.

[18] C. Komo, C. Beierle, Nonmonotonic reasoning from conditional knowledge bases with system W, Ann. Math. Artif. Intell. 90 (2022) 107–144.

[19] J. Heyninck, G. Kern-Isberner, T. Meyer, Lexicographic Entailment, Syntax Splitting and the Drowning Problem, in: L. D. Raedt (Ed.), Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI 2022, Vienna, Austria, 23-29 July 2022, ijcai.org, 2022, pp. 2662–2668. URL: https://doi.org/10.24963/ijcai.2022/369. doi:10.24963/ijcai.2022/369.

[20] J. Haldimann, C. Beierle, Inference with System W Satisfies Syntax Splitting, in: G. Kern-Isberner, G. Lakemeyer, T. Meyer (Eds.), 19th International Conference on Principles of Knowledge Representation and Reasoning, KR 2022, Haifa, Israel., 2022, pp. 405–409.

[21] C. Komo, C. Beierle, Nonmonotonic Inferences with Qualitative Conditionals Based on Preferred Structures on Worlds, in: U. Schmid, F. Klügl, D. Wolter (Eds.), KI 2020: Advances in Artificial Intelligence - 43rd German Conference on AI, volume 12325 of *LNCS*, Springer, 2020, pp. 102–115.

[22] J. Haldimann, C. Beierle, Properties of System W and Its Relationships to Other Inductive Inference Operators, in: I. Varzinczak (Ed.), Foundations of Information and Knowledge Systems - 12th International Symposium, FoIKS 2022, volume 13388 of *LNCS*, Springer, 2022, pp. 206–225.

[23] P. Marquis, N. Schwind, Lost in translation: Language independence in propositional logic–application to belief change, Artificial Intelligence 206 (2014) 1–24.

[24] E. Weydert, System JLZ–rational default reasoning by minimal ranking constructions, Journal of Applied Logic 1 (2003) 273–308.

[25] G. Kern-Isberner, J. Heyninck, C. Beierle, Conditional Independence for Iterated Belief Revision, in: L. D. Raedt (Ed.), Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI 2022, ijcai.org, 2022, pp. 2690–2696. URL: https://doi.org/10.24963/ijcai.2022/373. doi:10.24963/ijcai.2022/373.

[26] G. Casini, T. Meyer, K. Moodley, R. Nortjé, Relevant closure: A new form of defeasible reasoning for description logics, in: Logics in Artificial Intelligence: 14th European Conference, JELIA 2014, Funchal, Madeira, Portugal, September 24-26, 2014. Proceedings 14, Springer, 2014, pp. 92–106.

[27] R. Booth, I. Varzinczak, Conditional inference under disjunctive rationality, in: Proceedings of the AAAI Conference on Artificial Intelligence, volume 35, 2021, pp. 6227–6234.